A Statistical Proof of Chebyshev's Sum Inequality

Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be arbitrary real numbers satisfying $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots y_n$. The well-known Chebyshev's Sum Inequality states that

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right).$$

Typical proofs rely on clever algebra built upon the observation that the quantity $(x_i - x_j)(y_i - y_j)$ is nonnegative for all i, j. We offer a statistical proof. Let $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$. Since the x_i's and y_i's decrease together, they have nonnegative covariance, i.e.,

$$0 \leq \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = \frac{1}{n}\left(\sum_{i=1}^{n} x_i y_i - \overline{y}\sum_{i=1}^{n} x_i - \overline{x}\sum_{i=1}^{n} y_i + n\,\overline{x}\,\overline{y}\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i y_i - n\,\overline{x}\,\overline{y} - n\,\overline{x}\,\overline{y} + n\,\overline{x}\,\overline{y}\right) = \frac{1}{n}\left(\sum_{i=1}^{n} x_i y_i\right) - \overline{x}\,\overline{y}$$

Rearranging, we get $\frac{1}{n}\left(\sum_{i=1}^{n} x_i y_i\right) \geq \overline{x}\,\overline{y}$, as desired. If instead the real numbers satisfy $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \geq y_2 \geq \cdots y_n$, the covariance is nonpositive and the inequality is reversed.

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