Wald's Identity vs. Tail Sum Formula

Let N be an arbitrary nonnegative integer-valued random variable with finite expectation $\mathbb{E}[N]$. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed nonnegative random variables, independent of N, with $\mathbb{E}[X_1] < \infty$. Two well-known theorems in this context are Wald's Identity,

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}[X_1]\mathbb{E}[N],$$

and the Tail Sum Formula

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}(N \geq n).$$

We prove that these theorems are equivalent via the following calculation.

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}\left[\sum_{n=1}^{\infty} X_n \mathbb{I}\{N \geq n\}\right] = \sum_{n=1}^{\infty} \mathbb{E}[X_n \mathbb{I}\{N \geq n\}] =$$

$$= \sum_{n=1}^{\infty} \mathbb{E}[X_n]\mathbb{E}[\mathbb{I}\{N \geq n\}] = \mathbb{E}[X_1] \sum_{n=1}^{\infty} \mathbb{P}(N \geq n).$$

Here the first equality is justified because $\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n \mathbb{I}\{N \geq n\}$; the second because the X_i's are nonnegative; the third because N is independent of the X_i's; and the last because the X_i's are identically distributed.

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