INDETERMINATE EXPONENTIALS WITHOUT TEARS

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Every calculus student learns how to solve indeterminate limits of the form 1^∞; most quickly learn to hate and fear this process. It is error-prone, full of tedious algebra, and requires careful attention to L'Hôpital's rule. Here is a typical "fairly simple" example:

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{n+4}{n}\right)^{3n+1} = \lim_{n\to\infty} e^{\ln\left(\frac{n+4}{n}\right)^{3n+1}} = \lim_{n\to\infty} e^{\frac{\ln\left(\frac{n+4}{n}\right)}{\frac{1}{3n+1}}} &= e^{\lim_{n\to\infty}\frac{\ln\left(\frac{n+4}{n}\right)}{\frac{1}{3n+1}}} \\
&\overset{\text{L'H}}{=} e^{\lim_{n\to\infty}\frac{\frac{n}{n+4}\left(\frac{-4}{n^2}\right)}{\frac{-3}{(3n+1)^2}}} \\
&= e^{\lim_{n\to\infty}\frac{-4(3n+1)^2}{-3n(n+4)}} \\
&= e^{\lim_{n\to\infty}\frac{-36n^2-24n-4}{-3n^2-12n}} \\
&= e^{12}.
\end{aligned}$$

What tedium! And this is the short version, suppressing details on the two derivatives (perhaps two quotient rules, perhaps something slightly better). Of course, this may be tedious for students, but some people who are experts use simpler and shorter ways. Indeed, replacing n by $4k$ converts the limit to $\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^{12k+1}$, equivalently $\left(\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^{k}\right)^{12}$. So the problem reduces to the familiar limit.

Here, we are interested in formulating these methods as a general formula for calculating indeterminate limits of the form 1^∞. We prove the following theorem.

Theorem 0.1. *Suppose that $f(n)$ is a function with $\lim_{n\to\infty} f(n) = 1$, and $g(n)$ is a function with $\lim_{n\to\infty} g(n) = \infty$. Then*

$$\lim_{n\to\infty} f(n)^{g(n)} = e^{\lim_{n\to\infty} g(n)(f(n)-1)}.$$

We present two proofs for this theorem. In the first proof we assume that the function $f(n)$ is differentiable and then the L'Hôpital's rule is used. In the second proof needs neither L'Hôpital's rule, nor the hypothesis that $f(n)$ is differentiable, nor interpolation with cubic splines.

First proof. First we note that $\lim_{n\to\infty}\frac{\ln f(n)}{f(n)-1} \overset{\text{L'H}}{=} \lim_{n\to\infty}\frac{\frac{f'(n)}{f(n)}}{f'(n)} = \lim_{n\to\infty}\frac{1}{f(n)} = 1$. We start with the usual algebra, then finish with our observation:

$$\lim_{n\to\infty} f(n)^{g(n)} = e^{\lim_{n\to\infty} g(n)\ln f(n)} = e^{\lim_{n\to\infty} g(n)(f(n)-1)\frac{\ln f(n)}{f(n)-1}} = e^{\lim_{n\to\infty} g(n)(f(n)-1)}.$$

Second proof. Since $\ln(1+x) = x + o(x)$, equivalently $\lim_{n\to\infty}\frac{\ln(1+x)}{x} = 1$, replacing x with $f(n)-1$ shows that

$$\lim_{n\to\infty} f(n)^{g(n)} = \lim_{n\to\infty} e^{g(n)\ln(1+(f(n)-1))} = e^{\lim_{n\to\infty} g(n)(f(n)-1)}.$$

With Theorem 0.1 our "fairly simple"example becomes truly fairly simple:

$$\lim_{n\to\infty}\left(\frac{n+4}{n}\right)^{3n+1} = e^{\lim_{n\to\infty}(3n+1)\left(\frac{n+4}{n}-1\right)} = e^{\lim_{n\to\infty}\frac{4}{n}(3n+1)} = e^{\lim_{n\to\infty}12+\frac{4}{n}} = e^{12}.$$

Theorem 0.1 can be applied to the famous Euler's Limit $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$, and, to some extensions thereof, such as (from [1]) $\lim_{n\to\infty}\left(\frac{A_{n+1}}{A_n}\right)^{\frac{A_n}{A_{n+1}-A_n}} = e$ (if $A_{n+1}\sim A_n$).

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References

[1] R. Farhadian, A Generalization of Eulers Limit, *Amer. Math. Monthly.* **129** (2022), p. 384.

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