A Geometric mean–Arithmetic mean Ratio Limit

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One of the truly delightful results related to the natural numbers is the following limit of the ratio of the geometric and arithmetic means of the first n natural numbers:

$$\lim_{n\to\infty} \frac{\sqrt[n]{1\cdot 2\cdot 3\cdots n}}{\left(\frac{1+2+3+\cdots+n}{n}\right)} = \frac{2}{e}. \tag{1}$$

Obviously, the ratio in (1) approaches its limit really slowly. In fact, the relative difference between the ratio and its limiting value is of order, $(n+1)^{(2n)^{-1}}$, as $n \to \infty$. For example, this is about 2 percent when $n = 100$.

Some generalisation of the limit can be found in [1]–[3].

In this note, we offer a short proof and generalisation of limit (1). Our result is narrower here, but the techniques are wholly different from [1], [2], and [3], and rely solely, in theory, on algebraic limit properties. Our proof relies on the following well-known result.

Lemma. *[see, e.g., p.81 of [4]] Let a_n be a sequence of positive reals with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. Then $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.*

We now establish a generalisation of (1) in the following theorem.

Theorem. *Let $\{b_n\}$ be a sequence of positive reals with $\lim_{n\to\infty} b_n - n = 0$. Then*

$$\lim_{n\to\infty} \frac{\sqrt[n]{b_1 b_2 b_3 \cdots b_n}}{\left(\frac{b_1+b_2+b_3+\cdots+b_n}{n}\right)} = \frac{2}{e}.$$

Proof. We apply the lemma to $a_n = \left(\prod_{i=1}^n b_i\right)/\left(\frac{1}{n}\sum_{i=1}^n b_i\right)^n$. Note that $\frac{1}{n}\sum_{i=1}^n b_i = \frac{1}{n}\sum_{i=1}^n (b_i - i) + \frac{1}{n}\sum_{i=1}^n i$, and define $c_n = \frac{1}{n}\sum_{i=1}^n (b_i - i)$.

$$\frac{a_{n+1}}{a_n} = b_{n+1}\frac{\left(c_n + \frac{n+1}{2}\right)^n}{\left(c_{n+1} + \frac{n+2}{2}\right)^{n+1}} = \frac{b_{n+1}}{c_{n+1} + \frac{n+2}{2}}\left(\frac{n+1+2c_n}{n+2+2c_{n+1}}\right)^n.$$

Noting that $\lim_{n\to\infty} c_n = 0$, we see that the limit of the first part is 2. We may find the limit of the second part directly, or using the main result of [5]:

$$\lim_{n\to\infty}\left(\frac{n+1+2c_n}{n+2+2c_{n+1}}\right)^n = \exp\left(\lim_{n\to\infty} n\frac{-1+2c_n-2c_{n+1}}{n+2+2c_{n+1}}\right) = e^{-1}.$$

This completes the proof.

Note that taking $b_n = n$ in the theorem gives (1). As a general example, the theorem applies to any sequence $b_n = n + f(n)$, where $f(n) \to 0$. For example, $b_n = n + \frac{1}{\sqrt{n}}$.

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Reza Farhadian
Department of Statistics, Razi university, Kermanshah, Iran
Email: farhadian.reza@yahoo.com

Vadim Ponomarenko
Department of Mathematics and Statistics, San Diego State University, San Diego, USA
Email: vponomarenko@sdsu.edu