COMPLEMENTARY NUMERICAL SETS

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Abstract

A numerical set S is a cofinite subset of $\mathbb{N}$ which contains 0. We use the natural bijection between numerical sets and Young diagrams to define a numerical set $\widetilde{S}$, such that their Young diagrams are complementary to each other. We determine various properties of $\widetilde{S}$, particularly with an eye to closure under addition (for both S and $\widetilde{S}$), which allows a numerical set to become a numerical semigroup.

1. Introduction

We denote by $\mathbb{N}$ the set of non-negative integers. A *numerical set* is a cofinite subset of $\mathbb{N}$ that contains 0. The natural numbers that are missing from a numerical set S are called its *gaps*, and the collection of all gaps is denoted by $Gap(S)$. The largest gap is called the *Frobenius number* and is denoted by $F(S)$. The number of gaps is called the *genus* and is denoted by $g(S)$. By convention we set $F(\mathbb{N}) = -1$ and $g(\mathbb{N}) = 0$. The smallest non-zero element in S is called its *multiplicity* and is denoted by $m(S)$. A *numerical semigroup* is a numerical set that is closed under addition. Given a numerical set, there is a natural way of constructing a numerical semigroup from it, which is called the *atomic monoid* of the set or the *associated semigroup* of the set. We denote the associated semigroup of a numerical set S as $A(S)$, and it is given by

$$A(S) = \{s \in S \mid s + S \subseteq S\}.$$

It is straightforward to show that $A(S)$ is in fact a numerical semigroup and it has the same Frobenius number as S. Also note that if S was a numerical semigroup, then $A(S) = S$ since S is closed under addition. This operation of associated semigroup was defined by Antokoletz and Miller in [1] and has been studied in several recent papers [2, 3, 4, 5, 7, 9, 12]. See [11] for a general reference on numerical semigroups.

The *atoms* of a numerical semigroup S are the positive elements of S that cannot be written as a sum of two positive elements of S. The number of atoms of S is called its *embedding dimension* and is denoted by $e(S)$. It is known that $e(S) \leq m(S)$, and S is said to be of *max embedding dimension* if $e(S) = m(S)$. An atom (resp. positive element) of S is called a *small atom* (resp. *small element*) of S if it is smaller than the Frobenius number of S. Note that if S is a numerical set for which $A(S)$ has no small atoms then $A(S) = \{0, F(S) + 1 \rightarrow\}$. Here the $\rightarrow$ indicates that all numbers after $F(S) + 1$ are in the numerical set. Marzuola and Miller in [9] compute the density of such numerical sets among all numerical sets of a given Frobenius number. There are 2^{f-1} numerical sets with Frobenius number f. They prove that the following limit exists and is positive:

$$\lim_{f\to\infty} \frac{\#\{S \mid A(S) = \{0, f + 1 \rightarrow\}\}}{2^{f-1}} = \gamma.$$

The limit γ is approximately 0.48 meaning, for around 48% of the numerical sets $A(S)$ has no small atoms. The authors of [2] study the numerical sets for which $A(S)$ has one small element and they conjecture that for a fixed l the following limit exists and is positive:

$$\lim_{f\to\infty} \frac{\#\{S \mid A(S) = \{0, f-l, f + 1 \rightarrow\}\}}{2^{f-1}} = \gamma_l.$$

This conjecture is proved and extended to the case of n small elements by Singhal and Lin in [12]. If $A(S)$ has one small atom then $A(S)$ must be of the form $m\mathbb{N} \cup \{f+1 \rightarrow\}$. With m fixed, the authors of [4] enumerate the number of such numerical sets and show that it is a quasi-polynomial in f.

Kaplan et al. in [5] showed that numerical sets have a bijective correspondence to Young diagrams. A *Young diagram* is an array given by stacking rows of squares of varying length, but with the property that the rows are non-increasing in length as they progress down. An example is given in Figure 1. We now describe how to

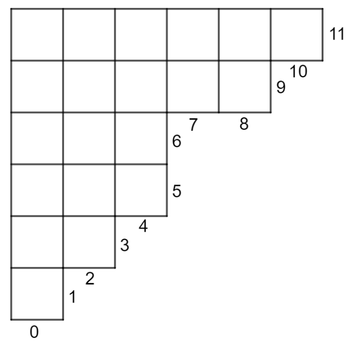

Figure 1: Young diagram corresponding to $S = \{0, 2, 4, 7, 8, 10, 12 \to\}$

get a Young diagram from a given numerical set. A Young diagram is determined by the path connecting its bottom left point to its top-right point. In order to get this path we start at the origin and consider the natural numbers starting from 0. We take a step right for every natural number that is in the numerical set and take a step upwards for every natural number not in the numerical set. We stop once we reach the Frobenius number. The path thus obtained will enclose a Young diagram. For example, the given Young diagram in Figure 1 would be obtained if our numerical set is $\{0, 2, 4, 7, 8, 10, 12 \to\}$. This process is reversible and it gives a one-to-one correspondence between numerical sets and Young diagrams.

The *complement* of a Young diagram is found by completing the rectangular grid with the length and width of the first row and first column respectively and then rotating the other piece by 180°. Look at Figure 2 for an example. We can start with a numerical set S and consider its Young diagram. Then take its complement and consider the numerical set associated with the complement. The numerical set thus obtained is called the *complement* of S and is denoted by $\widetilde{S}$. In our example with $S = \{0, 2, 4, 7, 8, 10, 12 \to\}$, we have $\widetilde{S} = \{0, 2, 3, 6, 8, 10 \to\}$. If S has the form $\{0, 1, 2, \ldots, n, F + 1 \to\}$ with $n < F$, then the Young diagram of S is already a rectangle. In this case, we set $\widetilde{S} = \mathbb{N}$. We leave $\widetilde{\mathbb{N}}$ undefined. It should be noted that applying the complement operation twice does not lead back to the original numerical set. For example, in Figure 2, it can be seen that $\widetilde{\widetilde{S}} \neq S$. Moreover, several numerical sets can have the same complement. In this paper, we continue the study of numerical semigroups with no small atoms, with one small atom, and with one small element, using the Young diagram tools described above. Our main results are as follows.

Theorem 2. *Given a numerical semigroup $S \neq \mathbb{N}$, $A(\widetilde{S})$ has at most one small atom. Moreover, if S has more than one small atom then $A(\widetilde{S})$ has no small atoms.*

Theorem 3. *If $S \neq \mathbb{N}$ is a numerical set for which $\widetilde{S}$ is a numerical semigroup,*

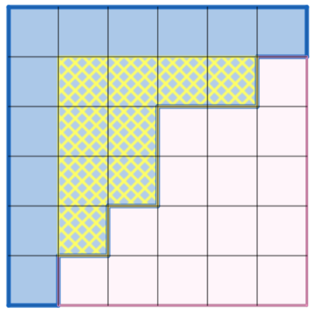

Figure 2: The Young diagram of numerical set S is in blue, that of $\widetilde{S}$ is in pink and the one for $\widetilde{\widetilde{S}}$ is in yellow.

then $A(S)$ has at most one small atom. Moreover, if S is not a numerical semigroup then $A(S)$ has at most one small element.

Corollary 1. *Given a numerical semigroup $S \neq \mathbb{N}$, its complement $\widetilde{S}$ is a numerical semigroup if and only if S has at most one small atom.*

Note that Theorem 2 and Theorem 3 cannot be obtained from each other by swapping S and $\widetilde{S}$. This is because $\widetilde{\widetilde{S}} \neq S$ (See Figure 2).

2. Structure of Complementary Numerical Sets

Young diagrams of numerical sets are closely related to their associated semigroups. Every box on the Young diagram has a *hook*, which consists of all the boxes below and to the right of the given box and that box itself. The number of boxes in the hook is called the *hook length* of that box. This is seen in Figure 3, where the number in each square is the hook length of that square, and a particular hook with hook length 5 is emphasized in green. Kaplan et al. [5] have shown that the hook lengths of a numerical set's Young diagram correspond precisely to the gaps of its associated semigroup.

There are often many numerical sets that have the same associated numerical semigroup which means that the set of hook lengths does not uniquely determine a Young diagram. Herman and Chung [6] found that the hook multi-set is also not unique to a particular Young diagram. They did, however, find that a numerical set is fully characterized by its hook multi-set accompanied by the hook multi-set of the complement of the Young diagram. We will now give a more direct description of the complement numerical set.

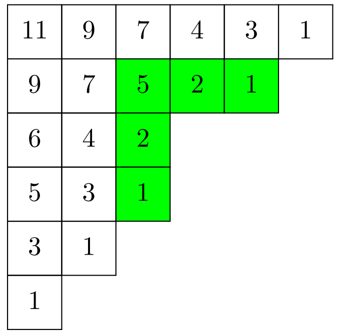

Figure 3: Since there are 2 squares to the right and 2 below, the hook length at this position is 5.

Definition 1. Let $S \neq \mathbb{N}$ be a numerical set. We define its *base* $B(S)$ as the biggest element in S smaller than the Frobenius number, namely

$$B(S) = \max\{s \in S \mid s < F(S)\}.$$

Theorem 1. *Let $S \neq \mathbb{N}$ be a numerical set with complement $\widetilde{S}$. Then,*

$$\tilde{S} = \{B(S) - s \mid s \in S \text{ and } s \leq B(S)\} \cup \{B(S) \rightarrow\}.$$

Proof. Consider the following Young diagram:

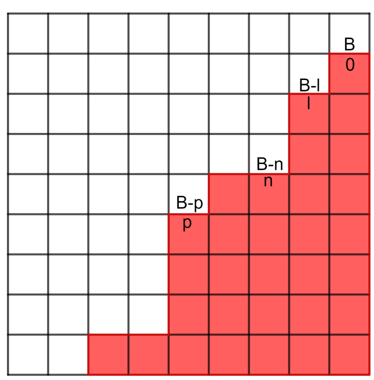

Figure 4: The Young diagrams of a numerical set and its complement

The 0 of $\widetilde{S}$ corresponds to $B(S)$ of S in the Young diagram. Following the path from there, we see that the l of $\widetilde{S}$ corresponds to $B(S) - l$ of S in the diagram as long as $0 \le l \le B(S)$. Since horizontal line segments remain horizontal upon a $180°$ rotation we see that for $0 \le l \le B(S)$, $l \in \widetilde{S}$ if and only if $B(S) - l \in S$. The 0 of S corresponds to $B(S)$ of $\widetilde{S}$, and the Young diagram of $\widetilde{S}$ finishes before this. This means that $[B(S), \infty) \subseteq \widetilde{S}$.

□

We now present some immediate properties of $\widetilde{S}$.

Proposition 1. *Let $S \neq \mathbb{N}$ be a numerical set with complement $\widetilde{S}$. Then:*
(a) $F(\widetilde{S}) \le B(S) - 1 \le F(S) - 2$ and further, if $1 \notin S$ then $F(\widetilde{S}) = B(S) - 1$;
(b) $g(\widetilde{S}) = g(S) + B(S) - F(S)$;
(c) $B(\widetilde{S}) \le B(S) - m(S)$ and, moreover, equality holds if and only if $1 \notin S$.

Proof. Parts (a) and (c) directly follow from Theorem 1. For part (b), Theorem 1 implies that $|Gap(S) \cap [0, B(S)]| = g(\widetilde{S})$. Moreover,

$$Gap(S) \cap [B(S) + 1, \infty) = [B(S) + 1, F(S)].$$

Therefore, $g(S) = g(\widetilde{S}) + (F(S) - B(S))$. □

We now characterize which numerical sets arise as the complements of numerical semigroups.

Proposition 2. *Let T be a numerical set. There exists a numerical semigroup $S \neq \mathbb{N}$ for which $T = \widetilde{S}$ if and only if*
(a) $F(T) \notin T + T$; and
(b) for every $x, y \in T \cap [0, F(T)]$ with $x + y > F(T)$, we have $x + y - F(T) - 1 \in T$.

Proof. First, suppose we have a numerical semigroup $S \neq \mathbb{N}$ for which $T = \widetilde{S}$. If there are $x, y \in T$ for which $x + y = F(T)$ then $x, y \le F(T) = B(S) - 1$. This means that $B(S) - x, B(S) - y \in S$, and since S is a numerical semigroup we have $2B(S) - x - y \in S$. However,

$$2B(S) - x - y = 2B(S) - (B(S) - 1) = B(S) + 1.$$

However, $B(S) < B(S) + 1 \le F(S)$, so $B(S) + 1$ cannot be in S and we have a contradiction.

Next, suppose $x, y \in T \cap [0, F(T)]$ with $x + y > F(T)$. We have $B(S) - x, B(S) - y \in S$ and hence $B(S) - (x + y - B(S)) \in S$. Further, since $0 \le x + y - B(S) \le 2F(T) - B(S) = B(S) - 2$, we get that $x + y - B(S) \in \widetilde{S} = T$ and of course $x + y - B(S) = x + y - F(T) - 1$.

Conversely if T is a numerical set that satisfies the two conditions then define

$$S = \{F(T) + 1 - x \mid x \in T \cap [0, F(T) + 1]\} \cup \{F(T) + 3 \rightarrow\}.$$

Clearly $F(S) = F(T) + 2$ (so $S \neq \mathbb{N}$), $B(S) = F(T) + 1$ and therefore $T = \widetilde{S}$. We need to show that S is a numerical semigroup. Given positive $a, b \in S$, if $a+b \geq F(T)+3$ then clearly $a+b \in S$. Otherwise $a+b \leq F(T)+2$, so $a, b \leq F(T)+1$ and hence $F(T) + 1 - a, F(T) + 1 - b \in T$. Now

$$2F(T) + 2 - a - b = F(T) + 1 - a + F(T) + 1 - b \neq F(T),$$

which means that $a+b \neq F(T)+2$ and hence $a+b \leq F(T)+1$. Let $x = F(T)+1-a$, $y = F(T) + 1 - b$ so $x, y \in T \cap [0, F(T)]$ and $x + y = 2F(T) + 2 - a - b \geq F(T) + 1$ and hence $x + y - F(T) - 1 \in T$. Finally $x + y - F(T) - 1 \leq F(T) + 1$ and hence $F(T) + 1 - (x + y - F(T) - 1) \in S$. Since $F(T) + 1 - (x + y - F(T) - 1) = a + b$, we see that S is indeed a numerical semigroup and we are done. $\square$

3. $A(\tilde{S})$ when S is a Numerical Semigroup

In this section, we prove Theorem 2. Toward this end, we consider a numerical semigroup $S \neq \mathbb{N}$ and study the associated semigroup of its complement. Note that if S has a single small atom then it must be the multiplicity $m(S)$. Since S is closed under addition it must also have all multiples of $m(S)$. However, if any of the elements in $S \cap [0, F(S)]$ is not a multiple of $m(S)$, then the smallest such element would be another small atom which is impossible. Thus if S is a numerical semigroup with a single small atom then $S \cap [0, F(S)]$ consists precisely of all the multiples of $m(S)$ that are in that range. On the other hand, if S has more than one small atom then $S \cap [0, F(S)]$ must contain elements that are not multiples of $m(S)$.

Theorem 2. *Given a numerical semigroup $S \neq \mathbb{N}$, $A(\widetilde{S})$ has at most one small atom. Moreover, if S has more than one small atom then $A(\widetilde{S})$ has no small atoms.*

Proof. Set $m = m(S)$ for convenience. If S has no small atoms, then $\widetilde{S} = \mathbb{N}$ is a numerical semigroup and has no small atoms. Next, if S has exactly one small atom then

$$S \cap [0, B(S)] = \{lm \mid 0 \leq l \leq N\},$$

where $B(S) = Nm$. Theorem 1 implies that $\widetilde{S} = m\mathbb{N} \cup \{Nm \rightarrow\}$, which is a numerical semigroup and has at most one small atom. Now for the rest of the proof assume that S has at least two small atoms.

Since $S \neq \mathbb{N}$ is a numerical semigroup we have $1 \notin S$, so by Theorem 1 we see that $F(\widetilde{S}) = B(S) - 1$ and also $B(\widetilde{S}) = B(S) - m$. We will first show that all elements in $A(\widetilde{S}) \cap [0, F(\widetilde{S})]$ must be multiples of m. Consider some $x \in \widetilde{S}$ with $1 \leq x < B(S) - 1$ and $m \nmid x$. Say $x = km + r$ with $1 \leq r \leq m - 1$. Now since

$(k+1)m \in S$ and $(k+1)m < B(S)$ (remember that $x \le B(\widetilde{S}) = B(S) - m$) we have $B(S) - (k+1)m \in \widetilde{S}$ by Theorem 1. Now

$$x + (B(S) - (k+1)m) = B(S) - (m - r).$$

Since $r \neq 0$ we know that $m - r \notin S$ and hence $B(S) - (m - r) \notin \widetilde{S}$. This shows that $x \notin A(\widetilde{S})$.

Now consider some $x \in \widetilde{S}$ of the form $x = lm$ for some $l \ge 1$, such that $x < B(S)$. Let nm be the largest multiple of m in $\tilde{S}$ such that $nm < B(S)$. We have two cases.

`Case 1.` $(n+1)m \notin \tilde{S}$. By Theorem 1, $B(S) - nm \in S$. Then since S is closed under addition, we have

$$B(S) - (n + 1 - l)m = B(S) - nm + (l - 1)m \in S.$$

This in turn implies that $(n + 1 - l)m \in \widetilde{S}$. Since

$$x + (n + 1 - l)m = (n + 1)m \notin \widetilde{S},$$

we see that $x \notin A(\widetilde{S})$.

`Case 2.` $(n+1)m \in \widetilde{S}$. By the definition of n, we get $(n+1)m \ge B(S) > nm$. Now since $nm \in \widetilde{S}$, we have $B(S) - nm \in S$. But $0 < B(S) - nm \le m$. Therefore, we must have $B(S) - nm = m$ i.e. $B(S) = (n + 1)m$.

Since S has at least two small atoms, there exists a $y \in S$ such that $m \nmid y$ and $y < F(S)$. Choose the smallest such y. Let $z = B(S) - y$ so z is the largest number in $\widetilde{S} \cap [0, B(S)]$ that is not a multiple of m. Say $z = jm + r'$ with $0 < r' < m$.

If $l \ge n - j$, then $y + (l - n + j)m \in S$ and hence

$$(n - l)m + r' = z - (l - n + j)m = B(S) - (y + (l - n + j)m) \in \widetilde{S}.$$

Now $x + (n - l)m + r' = nm + r'$. We know that $B(S) - (nm + r') = m - r' \notin S$ and hence $nm + r' \notin \widetilde{S}$. This means that $x \notin A(\widetilde{S})$.

Now consider the case when $l < n - j$. In this case

$$x + z = lm + jm + r' \le (n - 1)m + r' < nm < B(S).$$

This means that $x + z < B(S)$ and $x + z$ is not divisible by m. This means that $x + z \notin \widetilde{S}$ and hence $x \notin A(\widetilde{S})$. We conclude that $x \notin A(\widetilde{S})$ in all cases and hence $A(\widetilde{S})$ has no small atoms provided S has at least 2 small atoms. $\square$

During the proof we showed that if S has at most one small atom then $\widetilde{S}$ is a numerical semigroup. This proves one direction of Corollary 1.

4. $A(S)$ when $\widetilde{S}$ is a Numerical Semigroup

In this section, we will prove Theorem 3. We consider numerical sets $S \neq \mathbb{N}$ for which $\widetilde{S}$ is a numerical semigroup and study $A(S)$. We have already seen one such scenario in which S is a numerical semigroup with at most one small atom.

Theorem 3. *Let $S \neq \mathbb{N}$ be a numerical set for which $\widetilde{S}$ is a numerical semigroup. Then $A(S)$ has at most one small atom. Moreover, if S is not a numerical semigroup then $A(S)$ has at most one small element.*

Proof. Set $\widetilde{m} = m(\widetilde{S})$ for convenience. Let $b = Max(S \cap [0, B(S)-1])$. By Theorem 1 we know that $\widetilde{m} = B(S) - b$. We will first show that all elements of $A(S) \cap [0, B(S)-1]$ must be multiples of $\widetilde{m}$. Consider some $x \in S \cap [0, B(S)-1]$ that is not a multiple of $\widetilde{m}$. Say $x = q\widetilde{m}+r$ with $1 \leq r \leq \widetilde{m}-1$. We know that $(q+1)\widetilde{m} \in \widetilde{S}$ and $q\widetilde{m} \leq x \leq b = B(S) - \widetilde{m}$ which implies that $B(S) - (q+1)\widetilde{m} \in S$. Now

$$B(S) - (\widetilde{m} - r) = x + (B(S) - (q+1)\widetilde{m}).$$

However, $b = B(S) - \widetilde{m} < B(S) - (\widetilde{m} - r) < B(S)$ so $B(S) - (\widetilde{m} - r) \notin S$, which implies that $x \notin A(S)$.

Let $n\widetilde{m}$ be the largest multiple of $\widetilde{m}$ in $S \cap [0, B(S)-1]$. There are two cases.

`Case 1.` $(n+1)m(S) \notin S$. Suppose $x = q\widetilde{m} \in S$ with $0 < x \leq b$. We know that $B(S) - n\widetilde{m} \in \widetilde{S}$ so $B(S) - (n+1-q)\widetilde{m}$ is also in $\widetilde{S}$. This implies that $(n+1-q)\widetilde{m}$ is in S. We see that $x + (n+1-q)\widetilde{m} = (n+1)\widetilde{m}$ is not in S and hence $x \notin A(S)$. This implies that $A(S) \cap [1, B(S)-1] = \emptyset$.

`Case 2.` $(n+1)\widetilde{m} \in S$. The maximality of n implies that $(n+1)\widetilde{m} \geq B(S)$. This means that $B(S) - n\widetilde{m} \leq \widetilde{m}$, but we know that $B(S) - n\widetilde{m}$ is in $\widetilde{S}$ and it is positive. Therefore, $B(S) - n\widetilde{m} = \widetilde{m}$ i.e. $(n+1)\widetilde{m} = B(S)$. For $0 \leq j \leq n+1$, $(n+1-j)\widetilde{m} \in \widetilde{S}$ and hence $j\widetilde{m} = B(S) - (n+1-j)\widetilde{m} \in S$.

Now consider the case when all elements in $S \cap [0, B(S)]$ are multiples of $\widetilde{m}$. This implies that $m(S) = \widetilde{m}$. Now if $F(S) < (n+2)\widetilde{m} = B(S) + \widetilde{m}$ then it follows that S is a numerical semigroup and it has at most one small atom. On the other hand, if $B(S) + \widetilde{m} \leq F(S)$, then for each q with $1 \leq q \leq n+1$ we have $q\widetilde{m} + (n+2-q)\widetilde{m} = (n+2)\widetilde{m}$ which is not in S and hence $q\widetilde{m} \notin A(S)$. It follows that $A(S)$ has no small elements.

Finally consider the case when there are elements in $S \cap [0, B(S)]$ that are not multiples of $\widetilde{m}$. Let y be the largest such element, say $y = k\widetilde{m} + r'$ with $1 \leq r' \leq \widetilde{m} - 1$. Consider some $x = q\widetilde{m} \in S$ with $1 \leq q \leq n$.

Now if $q \leq n - k$ then

$$x + y \leq (n-k)\widetilde{m} + k\widetilde{m} + \widetilde{m} - 1 = B(S) - 1.$$

Moreover, $x + y \equiv r' \not\equiv 0(mod\ \widetilde{m})$, and therefore the maximality of y implies that $x + y \notin S$ and hence $x \notin A(S)$.

Otherwise, we have $q > n - k$. We know that $y \in S$ implies

$$(n + 1 - k)\widetilde{m} - r' = B(S) - y \in \widetilde{S}.$$

Since $\widetilde{S}$ is a numerical semigroup, we see that $(q+1)\widetilde{m} - r' \in \widetilde{S}$. Also $(q+1)\widetilde{m} - r' < (n + 1)\widetilde{m} = B(S)$, and this in turn tells us that

$$(n - q)\widetilde{m} + r' = B(S) - ((q + 1)\widetilde{m} - r') \in S.$$

Next, we have $x + (n - q)\widetilde{m} + r' = n\widetilde{m} + r'$. This quantity is strictly between b and $B(S)$ and hence it is not in S. Therefore, $x \notin A(S)$ and hence $A(S) \cap [1, B(S) - 1] = \emptyset$.

We have shown that S has at most one small atom and if S is not a numerical semigroup then $A(S) \cap [1, B(S) - 1] = \emptyset$. Assuming that S is not a numerical semigroup, we see that $A(S)$ is either $\{0, F(S) + 1 \rightarrow\}$ or $\{0, B(S), F(S) + 1 \rightarrow\}$, in particular, $A(S)$ has at most one small atom. $\square$

It is easy to see that $B(S) \in A(S)$ if and only if $S \cap [1, F(S) - B(S)] = \emptyset$. Therefore, given that $\widetilde{S}$ is a numerical semigroup and S is not, we see that if $[1, F(S) - B(S)] \cap S \neq \emptyset$ then $A(S) = \{0, F(S)+1 \rightarrow\}$ and if $[1, F(S) - B(S)] \cap S = \emptyset$ then $A(S) = \{0, B(S), F(S) + 1 \rightarrow\}$.

Corollary 1. *Given a numerical semigroup $S \neq \mathbb{N}$, its complement $\widetilde{S}$ is a numerical semigroup if and only if S has at most one small atom.*

Proof. We have already seen one direction: that if S is a numerical semigroup with at most one small atom then $\widetilde{S}$ is a numerical semigroup. Now for the other direction assume that S is a numerical semigroup for which $\widetilde{S}$ is also a numerical semigroup. Then by Theorem 3, we see that $A(S)$ has at most one small atom, but since $A(S) = S$ we are done. $\square$

5. Sequence of Complements

Given a numerical set S, we can construct a sequence of numerical sets by repeatedly applying the complement operation. The sequence is S, $\widetilde{S}$, $\widetilde{\widetilde{S}}, \ldots$. However, since $g(\widetilde{S}) < g(S)$, we know that the sequence would eventually reach $\mathbb{N}$. Denote $S^{(0)} = S$ and $S^{(i+1)} = \widetilde{S^{(i)}}$ provided $S^{(i)} \neq \mathbb{N}$. We will show that the length of this sequence is the number of boxes in the Young diagram of S that have hook length 1. Denote the number of boxes in the Young diagram of S with hook length 1 as $c_1(S)$.

Proposition 3. *Given a numerical set $S \neq \mathbb{N}$, we have $c_1(\widetilde{S}) = c_1(S) - 1$.*

Proof. In the Young diagram of S, boxes with hook length 1 appear at the corners (see Figure 5). Moreover, when one travels along the path between the Young diagrams of S and $\widetilde{S}$ and looks at the boxes with hook length 1, they alternately belong to S and $\widetilde{S}$. Since the first and last one both belong to S, it follows that $c_1(S) = c_1(\widetilde{S}) + 1$. $\square$

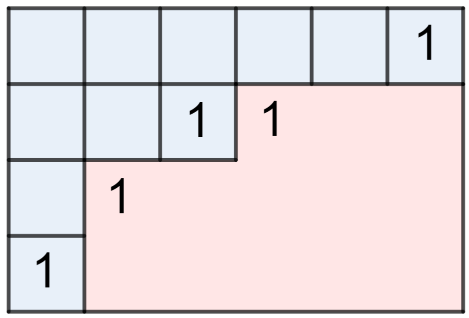

Figure 5: The Young diagrams of a numerical set and its complement with boxes of hook length 1 marked.

Corollary 2. *For any numerical set S, we have $S^{(c_1(S))} = \mathbb{N}$.*

Proof. For $n < c_1(S)$, we have $c_1(S^{(n)}) = c_1(S) - n > 0$ so $S^{(n)} \neq \mathbb{N}$. Moreover, $c_1(S^{(c_1(S))}) = 0$, so $S^{(c_1(S))} = \mathbb{N}$. $\square$

Proposition 4. *If S is a numerical set with $c_1(S) \geq 2$, then $A(S) \subseteq A(S^{(2)})$.*

Proof. The Young diagram of $S^{(2)}$ is obtained from the Young diagram of S by deleting several top rows and left columns. Therefore, the set of hook lengths of $S^{(2)}$ is a subset of hook lengths of S. As mentioned earlier, in [5] it was proved that the hook lengths of the Young diagram of S are precisely the gaps of $A(S)$. It follows that $A(S) \subseteq A(S^{(2)})$. $\square$

Proposition 2.12 of [11] gives a characterization of max embedding dimension numerical semigroups.

Lemma 1 ([11])**.** *A numerical semigroup S is of max embedding dimension if and only if $S' = \{x - m(S) \mid x \in S, x > 0\}$ is a numerical semigroup.*

Proposition 5. *Let S be a numerical semigroup with $c_1(S) \geq 2$. Then $S^{(2)}$ is a numerical semigroup if and only if $S \cup [B(S), \infty)$ is of max embedding dimension.*

Proof. Since S is a numerical semigroup with $c_1(S) \geq 2$, we know that $S \neq \mathbb{N}$. This means that $1 \notin S$ and hence $B(\widetilde{S}) = B(S) - m(S)$. It follows that

$$\begin{aligned} S^{(2)} &= \{B(\widetilde{S}) - a \mid a \in \widetilde{S}, a \leq B(\widetilde{S})\} \cup \{B(\widetilde{S}) \rightarrow\} \\ &= \{(B(\widetilde{S})) - (B(S) - x) \mid x \in S, x \leq B(S), B(S) - x \leq B(\widetilde{S})\} \cup \{B(\widetilde{S}) \rightarrow\} \\ &= \{x - m(S) \mid x \in S \cup [B(S), \infty), x > 0\}. \end{aligned}$$

Now Lemma 1 tells us that $S^{(2)}$ is a numerical semigroup if and only if $S \cup [B(S), \infty)$ has max embedding dimension. □

Corollary 3. *If S is a numerical semigroup of max embedding dimension with $c_1(S) \geq 2$ then $S^{(2)}$ is also a numerical semigroup.*

Proof. If S has max embedding dimension then Lemma 1 tells us that $S' = \{x - m(S) \mid x \in S, x > 0\}$ is a numerical semigroup and hence $S^{(2)} = S' \cup [B(S) - m(S), \infty)$ is also a numerical semigroup. □

Proposition 6. *Given a numerical semigroup S and a positive integer n, there exists another numerical semigroup T for which $T^{(2n)} = S$.*

Proof. We prove this for $n = 1$, the general case follows by induction. Let $T = \{0\} \cup \{x + m(S) \mid x \in S\} \setminus \{F(S) + 2m(S)\}$, then $m(T) = m(S)$, $F(T) = F(S) + 2m(S)$ and $B(T) = F(S) + 2m(S) - 1$. Next, if $x, y \in S$ then $x + y + m(S) \in S$ and $x + y \neq F(S)$ which means that $x + m(S) + y + m(S) \in T$. It follows that T is closed under addition and hence is a numerical semigroup. Finally

$$T \cup [B(T), \infty) = \{0\} \cup \{x + m(S) \mid x \in S\},$$

and hence $T^{(2)} = S$. □

Several other questions can be explored about the sequence of complements. For example,

1) If S is a numerical set and $S^{(2)}$ is a numerical semigroup then what can be said about $A(S)$?

2) More generally for a fixed $n \geq 2$, if S is a numerical set and $S^{(n)}$ is a numerical semigroup then what can be said about $A(S)$?

3) For a fixed odd $n \geq 3$, classify numerical semigroups S for which there is another numerical semigroup T with $T^{(n)} = S$.

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References

[1] E. Antokoletz and A. Miller, Symmetry and factorization of numerical sets and monoids, *J. Algebra* **247 (2)** (2002), 636-671.

[2] J.Arroyo, J. Autry, C. Crandall, J. Lefler and V. Ponomarenko, On numerical semigroups with almost-maximal genus, *PUMP J. Undergrad. Res.* **3** (2020), 62-67.

[3] A. Chen, C. O'Neill, D. Singhal and L. Lawson, Enumerating the associated numerical sets of a numerical semigroup (2020), unpublished manuscript.

[4] A. Chen, C. O'Neill, D. Singhal and L. Lawson, On the asymptotic growth of the number of associated numerical sets (2020), unpublished manuscript.

[5] H. Constantine, B. Houston-Edwards and N. Kaplan, Numerical sets, core partitions, and integer points in polytopes, in M. Nathanson, eds., *Combinatorial and Additive Number Theory II*, pp. 99-127, Springer-Cham, 2017.

[6] J. Herman and F. Chung, Some results on hook lengths, *Discrete Math.* **20** (1977), 33-40.

[7] W. Keith and R. Nath, Partitions with prescribed hooksets, *J. Comb. Number Theory* **3(1)** (2011), 39–50.

[8] B.F Logan and L.A Shepp, A variational problem for random Young tableaux, *Adv. Math.* **26 (2)** (1977), 206-222.

[9] J. Marzuola and A. Miller, Counting numerical sets with no small atoms. *J. Combin. Theory Ser. A* **117 (6)** (2010), 650-667.

[10] M. Omidali and F. Rahmati, On the type and the minimal presentation of certain numerical semigroups, *Comm. Algebra* (2009), 1275-1283.

[11] J.C. Rosales and P.A. Gracia-Sanchez, *Numerical Semigroups, Developments in Mathematics 20*, Springer 2009.

[12] D. Singhal and Y. Lin, Density of numerical sets associated to a numerical semigroup, *Comm. Algebra* **49 (10)** (2021), 4291-4303.