A Theorem on Indifference Graphs

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Abstract - Let P be a set of n points on the real line and let k be a fixed positive integer. Assume that for every $x \in P$ the set $\{y \in P \mid |y - x| \leq 1\}$ of all points in P at distance at most 1 from x has cardinality that is divisible by k. We show that necessarily n is divisible by k.

Keywords : indifference graph; interval graph; intersection graph; numerical monoid; numerical semigroup

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1 Introduction

An indifference graph is an undirected graph whose set of vertices is a finite multiset of real numbers, and whose edges are those pairs of vertices which are within distance one of each other, as real numbers. Since each vertex is within distance one of itself, we have a loop at each vertex. Indifference graphs are an important class of graphs with many applications to order theory and algebra among other areas, and are the subject of considerable study (see, e.g., [3, 4, 5, 6, 9, 10, 12, 14]).

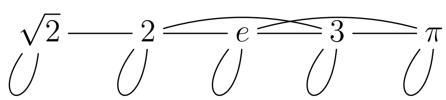

Figure 1: An indifference graph

For graph G, we denote by $n = |G|$ the *order* of $|G|$, i.e. the number of vertices of G. For vertex $v \in G$, we denote by $\deg(v)$ the *degree* of v, i.e. the number of neighbors of v in G, including v itself where we have a loop. Hence the example in Figure 1 has $\deg(\pi) = 3$.

Our main theorem is about indifference graphs where the degree of each vertex is divisible by a fixed positive integer k:

Theorem 1.1 *Let k be a fixed positive integer. Let G be an indifference graph. Assume that for every vertex v of G we have $k \mid \deg(v)$. Then necessarily $k \mid n$.*

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The case $k = 2$ in Theorem 1.1 was proved by Balof and appears in [8] in contrapositive form. This special case motivated our study in the current paper. For completeness, we provide the short standalone argument of the case $k = 2$ of Theorem 1.1 as it appears in [8]. Our proof of the general case in Theorem 1.1 is of different nature and thus provides a different (and slightly less elegant) proof of the case $k = 2$.

Theorem 1.2 (Balof in [8]) *Let G be an indifference graph. Assume that for every vertex v in G we have that $\deg(v)$ is even, then n is even.*

Proof. Consider the graph G' that is the same as G except that we remove all the loops from G. We thus get a simple graph and the degree of each vertex is now odd. It is a well known fact that every simple graph with an odd number of vertices must have a vertex whose degree is even (because the sum of the degrees of all vertices in every graph is even). Therefore, it must be that G', and therefore also G, cannot have an odd number of vertices. $\square$

The proof of Theorem 1.1 is given in Section 2. We remark that the analogue to Theorem 1.1 in two and higher dimensions is not true. To be more precise, one can define the indifference graph also for finite sets P of points in higher dimension. Two points in P form an edge iff they at distance at most 1 from each other. One can easily find even in the two-dimensional plane sets P of points such that the degree of every vertex in the indifference graph of P is divisible by some fixed positive integer k while $n = |P|$ is not divisible by k. To see this one can consider the set P of vertices of a regular n-gon in the plane circumscribed in a circle of some radius say 10, where n is a (not too small) prime. Then by symmetry the degrees of all vertices of the corresponding indifference graph are equal and denote by k their common value. Then $1 < k < n$ and because n is prime, then it is not divisible by k.

In Section 3 we suggest some further interesting problems in an attempt to generalize Theorem 1.1.

2 Proof of Theorem 1.1.

We prove the theorem by induction on $n = |G|$. The statement is clearly true for $0 \leq n \leq k$, since the hypothesis of $k|\deg(v)$ implies that n is either 0 or k, and the graph is complete. Assume $n > k$. Let G be an indifference graph with n vertices $x_1 \leq x_2 \leq \cdots \leq x_n$. Assume that for every vertex x of G the degree $\deg(x)$ is divisible by some fixed positive integer k. For every vertex x, we define $L(x)$ to be the vertex of smallest index (between 1, n) connected to x; we similarly define $R(x)$ to be the vertex of largest index connected to x. Note that L, R are nondecreasing functions. For vertices x_i, x_j with $1 \leq i < j \leq n$, we say that the interval $[x_i, x_j]$ is *right-inseparable* if there is no vertex x with $L(x) \in (x_i, x_j]$. We say $[x_i, x_j]$ is *left-inseparable* if there is no vertex x with $R(x) \in [x_i, x_j)$. We say that $[x_i, x_j]$ is *k-proper* if $k|(j - i + 1)$.

The following proposition is the key tool in the proof of the theorem.

Proposition 2.1 *There is a k-proper right-inseparable interval that is also left-inseparable.*

Proof. We first observe that there exists at least one k-proper right-inseparable interval, namely $[x_{n-k+1}, x_n]$. If it were not right-inseparable, then there would be some vertex x with $L(x) > x_{n-k+1}$. Then $L(x_n) \geq L(x) > x_{n-k+1}$, and consequently $\deg(x_n) < k$, a contradiction.

Now, let $[x_i, x_j]$ be the smallest (by length) k-proper right-inseparable interval. We claim that it is also left-inseparable. Suppose for the purpose of contradiction that it is not left-inseparable. Hence there is some vertex x with $R(x) = x_a \in [x_i, x_j)$. We first observe that $R(x_a) = R(x_{a+1})$ since otherwise we must have $L(R(x_{a+1})) = x_{a+1}$ violating the right-inseparable hypothesis on $[x_i, x_j]$. Next, observe that because $R(x) = x_a \in [x_i, x_j)$ we must have $L(x_a) \leq x < L(x_{a+1})$.

Choose vertex x_b maximal so that $x_b < L(x_{a+1})$. Therefore, $x_{b+1} = L(x_{a+1})$. Set $S = [L(x_a), x_b]$. Notice that S is k-proper. This is because the neighborhood of x_a is $[L(x_a), R(x_a)]$ and it contains the neighborhood of x_{a+1} which is $[x_{b+1}, R(x_a)]$. S is the set difference of these two neighborhoods. We notice further that S is right-inseparable. This is because otherwise there is some vertex y with $L(y) \in (L(x_a), x_b] \subset (L(x_a), L(x_{a+1}))$. By the monotonicity of the function L, this leads to $a < y < a + 1$, which is impossible. Finally, note that $x_i \leq x_a \leq L(x_a) + 1 < x_b + 1 < x_{a+1} \leq x_j$, so $|x_j - x_i| > |x_b - L(x_a)|$. Therefore, S is a k-proper right-inseparable interval of smaller length than $[x_i, x_j]$. This is a contradiction to the minimality of $[x_i, x_j]$. □

Proof of Theorem 1.1. By Proposition 2.1, we get vertices x_i, x_j, such that $[x_i, x_j]$ is a k-proper right-inseparable and also left-inseparable interval. Because $[x_i, x_j]$ is left-inseparable and right-inseparable, every vertex in G is either connected to every one of $\{x_i, x_{i+1}, \ldots, x_j\}$, or to none of them. Therefore, the indifference graph G' whose vertices are $\{x_1, \ldots, x_n\} \setminus [x_i, x_j]$ also satisfies that the degree is every vertex of G' is in $k\mathbb{N}_0$. By induction hypothesis $|G'|$ is divisible by k. The interval $[x_i, x_j]$ is k-proper meaning that $(j - i + 1)$ is divisible by k. Consequently, $|G| = |G'| + (j - i + 1)$ is also divisible by k, which completes the proof of the theorem. □

3 Further Problems and Concluding Remarks.

In this section, we suggest an interesting way of generalizing the result in Theorem 1.1.

We recall the notion of a *numerical monoid*. A numerical monoid is a subset of the nonnegative integers $\mathbb{N}_0$, containing 0, closed under addition. A numerical monoid that is also cofinite is called a numerical semigroup. Both of these are important objects (particularly numerical semigroups), themselves well-studied (see, e.g., [1, 2]). One simple example for a numerical monoid is the set $k\mathbb{N}_0 = \{km \mid m \in \mathbb{N} \cup \{0\}\}$, where k is a fixed integer.

Given a graph G and a subset S of the nonnegative integers, we say that G *respects* S if:

$$(\forall v \in G, \deg(v) \in S) \to |G| \in S.$$

Note that G can respect S vacuously, if the degree of some vertex of G does not appear in S. If instead all vertex degrees of G appear in S, then the order of G must also appear in S, for G to respect S.

By $[\![a,b]\!]$ we mean the integers between a and b, inclusive; similarly, by $[\![a,\infty)\!)$ we mean all integers greater than or equal to a. For any set $S \subseteq \mathbb{N}_0$, by kS we mean $\{kn : n \in S\}$. Note that the example in Figure 1 respects $2\mathbb{N}_0$ (vacuously), respects $[\![2,\infty)\!)$ and $[\![2,5]\!]$, but does not respect $[\![2,4]\!]$.

In the new terminology, a restatement of Theorem 1.1 is that every indifference graph respects $k\mathbb{N}_0$ for every positive integer k.

In the spirit of this restatement of Theorem 1.1 we raise the following two problems.

Problem 1 *Is it true that every indifference graph respects every numerical monoid?*

Problem 2 *Is it true that every indifference graph respects every numerical semigroup?*

Because a numerical semigroup is a numerical monoid, an affirmative answer to Problem 1 implies an affirmative answer to Problem 2. We observe that the converse is also true. An affirmative answer to Problem 2 implies an affirmative answer to Problem 1. This is because a graph G respect S if and only if it respects $S \cup [|G|+1,\infty)$ and notice that $S \cup [|G|+1,\infty)$ is already a numerical semigroup.

We note some basic observations in the following proposition.

Proposition 3.1 *Let G be a graph, S, S' be numerical monoids, and $a \in \mathbb{N}_0$.*

1. *Every indifference graph respects $0\mathbb{N}_0 = \{0\}$.*
2. *If G respects S, then G also respects $S \cup [\![a,\infty)\!)$.*
3. *If G respects S and S', then G also respects $S \cap S'$.*
4. *If G respects S, then G also respects $S \setminus [\![1,a+1]\!]$.*

Proof. (1) Every vertex in an indifference graph has degree at least one, since we assume it has a loop. Hence, all nonempty graphs respect $0\mathbb{N}_0$ vacuously, and the empty graph respects it nonvacuously.
(2) Suppose all vertices of G have their degree in $S \cup [\![a,\infty)\!)$. If any vertex v has $\deg(v) \notin S$, then $\deg(v) \geq a$; in this case, $|G| \geq a$ (looking at the neighbors of v alone), and so $|G| \in S \cup [\![a,\infty)\!)$. Otherwise, all vertices $v \in G$ have $\deg(v) \in S$. Since G respects S by hypothesis, we have $|G| \in S \subseteq S \cup [\![a,\infty)\!)$.
(3) Suppose all vertices of G have their degrees in $S \cap S'$. Then, in particular, all vertices of G have their degrees in S. Since G respects S, $|G| \in S$. Repeating for S' we find $|G| \in S'$; hence $|G| \in S \cap S'$.
(4) By (1) and (2), G respects $S' = \{0\} \cup [\![a+2,\infty)\!)$; now apply (3). □

Combining Theorem 1.2 and 3.1 we get the following result.

Corollary 3.2 *Let $t \in \mathbb{N}$ be odd. Every indifference graph respects the numerical semigroup $\langle 2,t\rangle = \{2x+ty : x,y \in \mathbb{N}_0\}$.*

Proof. $\langle 2,t\rangle = 2\mathbb{N}_0 \cup [\![t,\infty)\!)$. □

We can move from one numerical monoid to another with the following observation.

Theorem 3.3 *Let S be a numerical monoid and k a positive integer. Suppose that every indifference graph respects kS. Then every indifference graph respects S.*

Proof. Let G be an indifference graph with $\deg(v) \in S$ for all $v \in G$. We produce a new indifference graph G' which has the same vertices as G, except that every vertex is repeated k times. Hence $\deg(v) \in kS$ for all $v \in G'$. Since G' respects kS, we have $|G'| \in kS$. Hence $|G| \in S$. □

In particular, Theorem 3.3 tells us that if a numerical monoid T is respected by all indifference graphs, we can set $k = \gcd(T)$. Then $T = kS$ for a numerical monoid S with $\gcd(S) = 1$, and now all indifference graphs respect S. It is well-known (e.g. [1]) that a numerical monoid with no common factor is a numerical semigroup, so in fact S is a numerical semigroup.

We turn briefly to numerical semigroups. Given a numerical semigroup S, its *multiplicity* $m(S)$ is the smallest positive integer contained in S. Its *Frobenius number* $F(S)$ is the largest integer not contained in S. Let $\mathfrak{S}$ denote the set of numerical semigroups S such that $2m(S) > F(S)$. This set $\mathfrak{S}$ is independently interesting (see, e.g. [7, 11]). Further, $\mathfrak{S}$ has been shown (in [13]) to be asymptotically a strictly positive fraction of all numerical semigroups.

Theorem 3.4 *Let $S \in \mathfrak{S}$. Then every indifference graph respects S.*

Proof. Let G be an indifference graph with $\deg(v) \in S$ for all $v \in G$. Set $n = |G|$, and choose a, b such that $R(x_1) = x_a$ and $L(x_n) = x_b$. Since every vertex degree must be at least $m(S)$, we have $a \geq m(S)$ and $b \leq n - m(S) + 1$. We have two cases: if $b > a$, then $n - m(S) + 1 > m(S)$ and hence $n \geq 2m(S) > F(S)$. Since $n > F(S)$ and $F(S)$ is the last integer missing from S, we must have $n \in S$. If instead $b \leq a$, then x_a is connected to all vertices, namely $\deg(x_a) = n$, and hence in this case also $n \in S$. □

Our last observation is that indifference graphs of sets of small diameter, respect every numerical monoid.

Theorem 3.5 *Let G be an indifference graph with vertices $x_1 < \ldots < x_n$. Suppose that G has diameter is at most 2, that is $x_n - x_1 \leq 2$. Then G respects every numerical monoid S.*

Proof. Let S be a numerical monoid and suppose that for every vertex v of G we have $\deg(v) \in S$. In the notation of the proof of Theorem 1.1, if $R(x_1) < L(x_n)$, then there is no common neighbor to x_1 and x_n. On the other hand every vertex in G is at distance at most 1 from either x_1, or x_n. Therefore, $|G| = \deg(x_1) + \deg(x_n) \in S$.

It remains to consider the case $R(x_1) \geq L(x_n)$. In this case every vertex in G is a neighbor of $R(x_1)$ and therefore $|G| = \deg(R(x_1)) \in S$. □

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