Math 254 Exam 6 Solutions

1. Carefully state the definition of "basis". Give two examples, each from $\mathbb{R}^1$.

   A set of vectors is a basis if it is spanning and linearly independent. *Every* nonzero vector from $\mathbb{R}^1$, in a set by itself, is a basis. e.g. $\{(1)\}, \{(3)\}, \{(-\pi)\}$. Note that "3" is not a vector, and "(3)" is a vector, but not a set of vectors; neither of these is a basis.

2. Consider the vector space $\mathbb{R}[x,y]$ of polynomials in x, y. Let V be the subspace generated by basis $\{1, x, y, x^2, xy, y^2\}$. Set $S = \{(x+1)^2, (y-1)^2, (x+1)(y-1), (x+y)^2\}$. Determine if S is linearly independent.

   We must choose some order the basis of V, e.g. $v_1 = 1, v_2 = x, v_3 = y, v_4 = x^2, v_5 = xy, v_6 = y^2$. Then, $s_1 = x^2+2x+1 = v_1 + 2v_2 + v_4, s_2 = y^2 - 2y + 1 = v_1 - 2v_3 + v_6, s_3 = xy + y - x - 1 = -v_1 - v_2 + v_3 + v_5, s_4 = x^2 + 2xy + y^2 = v_4 + 2v_5 + v_6$. We now put these in either rows or columns (our choice), and do row reduction. $\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Since there are three pivots but $|S| = 4$, S is linearly dependent.

3. In the vector space $\mathbb{R}^2$, set $S = \{(2,3), (5,7)\}$, a basis. Find the change-of-basis matrix from the standard basis to S, and use this matrix to find $[(7,10)]_S$.

   We have $P_{ES} = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$, but we want $P_{SE} = P_{ES}^{-1} = \begin{pmatrix} -7 & 5 \\ 3 & -2 \end{pmatrix}$, which we can either calculate or remember the formula for. $[(7,10)]_S = P_{SE}[(7,10)]_E = \begin{pmatrix} -7 & 5 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4. In the vector space $\mathbb{R}^3$, set $S = \{(2,3,1), (0,1,2), (3,4,0)\}$, a basis. Find the change-of-basis matrix from the standard basis to S, and use this matrix to find $[(7,11,3)]_S$.

   $P_{ES} = \begin{bmatrix} 2 & 0 & 3 \\ 3 & 1 & 4 \\ 1 & 2 & 0 \end{bmatrix}$. We calculate P_{ES}^{-1} via $\begin{bmatrix} 2 & 0 & 3 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & -4 & 3 & 1 & 0 & -2 \\ 0 & -5 & 4 & 0 & 1 & -3 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & -5 & 4 & 0 & 1 & -3 \\ 0 & -4 & 3 & 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -0.8 & 0 & -0.2 & 0.6 \\ 0 & -4 & 3 & 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -0.8 & 0 & -0.2 & 0.6 \\ 0 & 0 & -0.2 & 1 & -0.8 & 0.4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -0.8 & 0 & -0.2 & 0.6 \\ 0 & 0 & 1 & -5 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & -5 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 8 & -6 & 3 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & -5 & 4 & -2 \end{bmatrix}$. Hence $[(7,11,3)]_S = \begin{bmatrix} 8 & -6 & 3 \\ -4 & 3 & -1 \\ -5 & 4 & -2 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

5. In the vector space $M_{2,2}(\mathbb{R})$, the set of 2×2 matrices, set $A = \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 4 & -2 \\ -9 & 5 \end{pmatrix}$. Set $S = Span\{A, B, C\}$. Find the dimension of S, and a basis.

   We must first choose an order for the standard basis for $M_{2,2}$. I choose $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We now put A, B, C in terms of E, as either rows or columns (our choice), and do row reduction.

   Rows: $\begin{bmatrix} 2 & 0 & -3 & 1 \\ 1 & 1 & 0 & -1 \\ 4 & -2 & -9 & 5 \end{bmatrix} \to \begin{bmatrix} 0 & -2 & -3 & 3 \\ 1 & 1 & 0 & -1 \\ 0 & -6 & -9 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -2 & -3 & 3 \\ 0 & -6 & -9 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & -2 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Hence $dim(S) = 2$ and a basis for S is given by the nonzero rows, namely $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ -3 & 3 \end{pmatrix} \right\}$.

   Columns: $\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & -2 \\ -3 & 0 & -9 \\ 1 & -1 & 5 \end{bmatrix} \to \begin{bmatrix} 0 & 3 & -6 \\ 0 & 1 & -2 \\ 0 & -3 & 6 \\ 1 & -1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Hence $dim(S) = 2$. Since the pivots are in columns 1,2, a basis for S is given by the first and second elements of S, namely $\{A, B\}$.