Math 254 Fall 2012 Exam 2a Solutions

1. Carefully state the definition of "degenerate" in the context of linear combinations. Give two examples.

   A linear combination is degenerate if all its coefficients are zero. Many examples are possible, such as $0, 0x, 0x + 0y, 0x_1 + 0x_2 + 0x_3$.

2. Solve the following system, using back-substitution. Be sure to justify your calculations.

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 + x_4 &= 6 \\ 4x_2 - 2x_3 + x_4 &= 10 \\ 5x_3 + 5x_4 &= 5 \\ 2x_4 &= 8 \end{aligned}$$

   $2x_4 = 8$, hence $x_4 = 4$. $5x_3 + 20 = 5$, hence $5x_3 = -15$ and $x_3 = -3$. $4x_2 + 6 + 4 = 10$, hence $4x_2 = 0$ and $x_2 = 0$. $4x_1 + 0 - 6 + 4 = 6$, hence $4x_1 = 8$ and $x_1 = 2$. Combining, there is a unique solution $(x_1, x_2, x_3, x_4) = (2, 0, -3, 4)$.

3. Consider the system of equations $\{3x - 2y = 1, kx + 4y = -2\}$. For which values of k (if any) does this have exactly one solution (and what is it)? For which values of k (if any) does this have no solution? For which values of k (if any) does this have infinitely many solutions?

   Replacing Eq.1 by (2 Eq.1+Eq.2), we get $(6 + k)x + 0y = 0$. If $k \neq -6$, then this implies $x = 0$; substituting into Eq.2 we get $4y = -2$ and $y = -0.5$. So, for $k \neq -6$, there is exactly one solution: $(0, -0.5)$. However, if $k = -6$, then we get infinitely many solutions: $(x, \frac{3x-1}{2})$, for every x. 'No solutions' cannot occur, since we have considered all possible values for k.

4. Find the line of best fit for the following set of points: $\{(2, 0), (1, -1), (0, 4)\}$.

   We calculate $N = 3, \sum x_i = 2 + 1 + 0 = 3, \sum x_i^2 = 4 + 1 + 0 = 5, \sum y_i = 0 - 1 + 4 = 3, \sum x_i y_i = 0 - 1 + 0 = -1$. Our system to solve is $\{3b + 3m = 3, 3b + 5m = -1\}$. Eq.2-Eq.1 is $2m = -4$, so $m = -2$. Plugging into Eq.1 we get $3b - 6 = 3$, so $b = 3$. Hence $y = -2x + 3$ is the desired line.

5. Solve the following system of linear equations using Gaussian elimination and back-substitution.

$$\begin{aligned} 2x + y + 2z &= 1 \\ -4x + 3z &= 1 \\ 6x - 2y - 3z &= 4 \end{aligned}$$

   We begin with Eq.2→Eq.2 + 2 Eq.1, Eq.3→Eq.3 - 3 Eq.1. This gives $\begin{aligned} 2x + y + 2z &= 1 \\ 0 + 2y + 7z &= 3 \\ 0 - 5y - 9z &= 1 \end{aligned}$. We now do Eq.3→2Eq.3 +5 Eq.2, to get $\begin{aligned} 2x + y + 2z &= 1 \\ 0 + 2y + 7z &= 3 \\ 0 + 0 + 17z &= 17 \end{aligned}$. Now we can do back-substitution: $17z = 17$, so $z = 1$; $2y + 7 = 3$, so $y = -2$; $2x - 2 + 2 = 1$, so $x = 0.5$. Hence there is a unique solution $(x, y, z) = (0.5, -2, 1)$.