Math 254 Fall 2012 Exam 2b Solutions

1. Carefully state the definition of "linear function". Give two examples.

   A linear function is a function, in one or more variables, that consists entirely of some mixture of addition and multiplication by constants, and no constant is added by itself. Many examples are possible, such as $f(x) = x, g(x, y) = 2x + 3y, h(x, y, z) = 3x - 4z$.

2. What is partial pivoting and why would you use it?

   Partial pivoting is a strategy used during Gaussian elimination, where each pivot is chosen to have maximal absolute value among all the candidates in that column. Its purpose is to improve stability (i.e. reduce roundoff error) in the calculation.

The remaining problems all concern the following system:
$$\begin{array}{rcl} 2x_1+3x_2+4x_4 &=& 9 \\ 4x_1+x_2-10x_3+3x_4 &=& -2 \\ -2x_1+2x_2+10x_3+2x_4 &=& 12 \\ -2x_1+x_2+8x_3+2x_4 &=& 9 \\ 4x_1+3x_2-6x_3+x_4 &=& 2 \end{array}$$

3. Write the system as an augmented matrix. Put this in echelon form, justifying each step. Use this to find the general solution to the system.

   $$\begin{bmatrix} 2 & 3 & 0 & 4 & 9 \\ 4 & 1 & -10 & 3 & -2 \\ -2 & 2 & 10 & 2 & 12 \\ -2 & 1 & 8 & 2 & 9 \\ 4 & 3 & -6 & 1 & 2 \end{bmatrix} \begin{array}{c} R_2-2R_1 \\ R_3+R_1 \\ R_4+R_1 \\ R_5-2R_1 \end{array} \longrightarrow \begin{bmatrix} 2 & 3 & 0 & 4 & 9 \\ 0 & -5 & -10 & -5 & -20 \\ 0 & 5 & 10 & 6 & 21 \\ 0 & 4 & 8 & 6 & 18 \\ 0 & -3 & -6 & -7 & -16 \end{bmatrix} (1/5)R_2 \longrightarrow \begin{bmatrix} 2 & 3 & 0 & 4 & 9 \\ 0 & -1 & -2 & -1 & -4 \\ 0 & 5 & 10 & 6 & 21 \\ 0 & 4 & 8 & 6 & 18 \\ 0 & -3 & -6 & -7 & -16 \end{bmatrix} \begin{array}{c} R_3+5R_2 \\ R_4+4R_2 \\ R_5-3R_2 \end{array} \longrightarrow$$
   $$\begin{bmatrix} 2 & 3 & 0 & 4 & 9 \\ 0 & -1 & -2 & -1 & -4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -4 & -4 \end{bmatrix} \begin{array}{c} R_4-2R_3 \\ R_5+4R_3 \end{array} \longrightarrow \begin{bmatrix} 2 & 3 & 0 & 4 & 9 \\ 0 & -1 & -2 & -1 & -4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
   We employ back-substitution. $x_4 = 1$, x_3 is free, $-x_2 - 2x_3 - x_4 = -4$ so $x_2 = 3 - 2x_3$, and $2x_1 + 3x_2 + 4x_4 = 9$, hence $2x_1 + 3(3 - 2x_3) + 4 = 9$ and so $x_1 = -2 + 3x_3$. Hence $(-2 + 3x_3, 3 - 2x_3, x_3, 1)$ is the general solution.

4. Beginning with the echelon form from the previous problem, put the augmented matrix in row canonical form, justifying each step. Use this to find the general solution to the system.

   $$\begin{bmatrix} 2 & 3 & 0 & 4 & 9 \\ 0 & -1 & -2 & -1 & -4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1+3R_2 \longrightarrow \begin{bmatrix} 2 & 0 & -6 & 1 & -3 \\ 0 & -1 & -2 & -1 & -4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} R_1-R_3 \\ R_2+R_3 \end{array} \longrightarrow \begin{bmatrix} 2 & 0 & -6 & 0 & -4 \\ 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} (1/2)R_1 \\ -R_2 \end{array} \longrightarrow$$
   $$\begin{bmatrix} 1 & 0 & -3 & 0 & -2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
   Now back-substitution is easy, yielding the same solution as in (3).

5. Solve the related homogeneous system (you may use your previous work). Use this solution and the particular solution $(1, 1, 1, 1)$ to give the general solution to the original system.

   The homogeneous system arises by replacing the last column of the augmented matrix with 0's. The same elementary row operations as above will yield $\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Again x_3 is free, and $x_1 = 3x_3, x_2 = -2x_3, x_4 = 0$. Hence the general solution to the homogeneous system is $(3a, -2a, a, 0)$ and to the original system is $(1, 1, 1, 1) + (3a, -2a, a, 0) = (1 + 3a, 1 - 2a, 1 + a, 1)$.