Math 254 Fall 2013 Exam 6 Solutions

1. Carefully state the definition of "standard vector space". Give a vector from $\mathbb{R}^3$ and a dependent set of two vectors from $\mathbb{R}^4$.

   The standard vector space $\mathbb{R}^n$ consists of the set of all ordered n-tuples of real numbers. $(1,2,3) \in \mathbb{R}^3$ and $\{(1,1,2,3),(2,2,4,6)\}$ is a dependent set of two vectors from $\mathbb{R}^4$.

2. Suppose that V is an n-dimensional vector space with basis S. Suppose that $\{v_1, v_2, \dots, v_k\}$ is independent in V. Prove that $\{[v_1]_S, [v_2]_S, \dots, [v_k]_S\}$ is independent in $\mathbb{R}^n$.

   Suppose we have a linear combination in $\mathbb{R}^n$ yielding zero: $a_1[v_1]_S + a_2[v_2]_S + \cdots + a_k[v_k]_S = \overline{0}$. Using the isomorphism properties, we have $a_1[v_1]_S + a_2[v_2]_S + \cdots + a_k[v_k]_S = [a_1v_1]_S + [a_2v_2]_S + \cdots + [a_kv_k]_S = [a_1v_1 + a_2v_2 + \cdots a_kv_k]_S = [0]_S$. Since $[\cdot]_S$ is one-to-one, $a_1v_1 + a_2v_2 + \cdots a_kv_k = 0$ in V. Since $\{v_1, v_2, \dots, v_k\}$ is independent, in fact $a_1 = a_2 = \cdots = a_k = 0$; hence the original linear combination was necessarily degenerate.

3. In the vector space $\mathbb{R}^2$, set $S = \{(1,3),(2,5)\}$, a basis. Find the change-of-basis matrix from the standard basis E to S, and use this matrix to find $[(1,1)]_S$.

   We find P_{ES} by writing S as columns; i.e. $P_{ES} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$. We want $P_{SE} = P_{ES}^{-1}$, which we can find via formula or using the $(A|I) \to (I|A^{-1})$ algorithm. $P_{SE} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$. Lastly, $[(1,1)]_S = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

The remaining two questions concern the vector space $P_3(t)$. Let $v_1 = -t^3 + t^2 + t + 1$, $v_2 = t^3 + 2t^2 + 2t + 1$, $v_3 = 4t^3 + 3t^2 + 3t + 1$, $v_4 = 4t^3 + 4t^2 + 4t + 2$.

4. Find a basis for $S = Span(\{v_1, v_2, v_3, v_4\})$.

   We use for coordinates the standard basis $E = \{1, t, t^2, t^3\}$, then put the result as rows in a matrix as $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 2 & 1 \\ 1 & 3 & 3 & 4 \\ 2 & 4 & 4 & 4 \end{pmatrix}$. We row reduce as $R_2 - R_1 \to R_2, R_3 - R_1 \to R_3, R_4 - 2R_1 \to R_4$, which gives $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 5 \\ 0 & 2 & 2 & 6 \end{pmatrix}$. $R_3 - 2R_2 \to R_3, R_4 - 2R_2 \to R_4$, which gives $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Lastly, $R_4 - 2R_3 \to R_4$ to get $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. We can now read off the basis from the rows, then translate back to $P_3(t)$, as $\{-t^3+t^2+t+1, 2t^3+t^2+t, t^3\}$.

5. Let $u = t^3 + t^2 + t + 1$. Determine whether or not $u \in S = Span(\{v_1, v_2, v_3, v_4\})$.

   We can append a row to the matrix from (4), or directly using inspection with the basis we found: $u = 1(-t^3 + t^2 + t + 1) + 2(t^3)$. (the answer is YES)

Extra: Consider the vector space $M_{2,2}$. Set $S = \left\{\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}\right\}$. Determine whether or not $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \in Span(S)$.

We use for coordinates the standard basis $E = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$, then put the results as rows in a matrix as $\begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 4 \\ 0 & 2 & 2 & 4 \end{pmatrix}$. This has row canonical form $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. We now include the vector $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ as a new row, giving matrix $\begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 4 \\ 0 & 2 & 2 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}$. This has row canonical form $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, which is different (higher rank), so the answer is NO.