MATH 254: Introduction to Linear Algebra
Chapter 0: Fundamental Definitions of Linear Algebra

Behold the most important ideas of the course, in bold. Please *memorize* them; they will be tested on every exam. Further, you need to *understand* them in all their intricacies – you should be able to provide examples and determine whether an object you are given meets a particular definition or not. A definition is a sentence and must satisfy all ordinary rules of English grammar. Generally each noun, verb, and adjective in a definition is *essential* and omitting even one of these would not be correct.

1. A vector space is a collection of objects called vectors, together with a way to add vectors and multiply by real numbers (called scalars). These latter properties are together called *closure*; two equivalent statements are given in the comments. We normally denote the vector space with upper case letters like V, U, W, and the vectors themselves with lower case letters like u, v, v_1, v'. Sometimes to emphasize that they are vectors we will put a bar or arrow over the top as $\overline{u}$ or $\vec{u}$.

2. For any set of vectors $\{v_1, v_2, \ldots, v_k\}$, their span is the set $\{a_1v_1 + a_2v_2 + \cdots + a_kv_k\}$, where each of $a_1, a_2, \ldots, a_k$ varies over every real number. Note that the span is also a set of vectors, which is a subset of the vector space from which the original set is drawn. We denote it as $Span(v_1, v_2, \ldots, v_k)$, and call the elements of this set linear combinations of $v_1, v_2, \ldots, v_k$. More compactly, we write $Span(v_1, v_2, \ldots, v_k) = \{a_1v_1 + a_2v_2 + \cdots + a_kv_k : a_1, a_2, \ldots, a_k \in \mathbb{R}\} = \{\sum_{i=1}^k a_iv_i : a_i \in \mathbb{R}\}$.

The next five definitions are the most important examples of vector spaces, at least in this course.

3. The linear function space in a set of variables $\{x_1, x_2, \ldots, x_k\}$ is their span, or (using the above notation) $Span(x_1, x_2, \ldots, x_k)$. Note: the vectors in this vector space are linear functions, such as $3x$ or $4x - 2y$. Note that a linear function may NOT include a constant, e.g. $4x + 5y + 3$ is not linear.

4. The polynomial space in a variable t, denoted $P(t)$, is the set of all polynomials in the single variable t. Note: the vectors in this vector space are polynomials, like $2 + t$ or $3 + 7t - 4t^5$. Often we prefer a subset of this space, by restricting to a maximum degree n, which we denote $P_n(t)$. For example, $6t^2 + 3t - 4$ and $-4t^2 + 8$ are both in $P(t)$, and also $P_2(t), P_3(t), \ldots$. Neither is in $P_1(t)$ or $P_0(t)$.

5. For any positive integers m, n, the matrix space $M_{m,n}$ is the set of all matrices with m rows and n columns (with real numbers as entries). Note: the vectors in this vector space are matrices, like $\left(\begin{smallmatrix} 1 & 2 \\ 3 & 5 \end{smallmatrix}\right)$. If $m = n$ we say the matrix is *square*, and sometimes abbreviate $M_{n,n}$ as M_n.

6. For any positive integer n, the standard vector space $\mathbb{R}^n$ is the set of all n-tuples of real numbers. Note: the vectors in this vector space are lists of n numbers, like $(1, 2)$ or $(4, 5, 6)$. These lists do not have an inherent orientation and may be written as convenient. A simple and pleasant example is with $n = 2$, namely $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$, because it's easy to draw vectors as arrows on the Cartesian plane.

7. For any set of vectors $\{v_1, v_2, \ldots, v_k\}$ drawn from vector space V, we say that this set is spanning if $Span(v_1, v_2, \ldots, v_k) = V$. We know that $Span(v_1, v_2, \ldots, v_k) \subseteq V$ holds for any set of vectors, so $\{v_1, v_2, \ldots, v_k\}$ is spanning if $Span(v_1, v_2, \ldots, v_k) \supseteq V$ also holds.

8. For any set of vectors $\{v_1, v_2, \ldots, v_k\}$, their nondegenerate span is the set $\{a_1v_1 + a_2v_2 + \cdots + a_kv_k\}$, where each of $a_1, a_2, \ldots, a_k$ varies over every real number *except* $a_1 = a_2 = \cdots = a_k = 0$. Note that the regular span will always contain the vector 0, but the nondegenerate span may or may not contain 0.

9. For any set of vectors $\{v_1, v_2, \ldots, v_k\}$, we say that this set is dependent if their nondegenerate span contains the vector 0. Otherwise, we say this set is independent; i.e. if their nondegenerate span does not contain the vector 0.

10. For any set of vectors $\{v_1, v_2, \ldots, v_k\}$ drawn from vector space V, we say that this set is a basis for V if it is both spanning and independent.

Comments on the Definitions:

1. Vector space closure in V can be expressed in either of the following two ways:

   Closure 1: For every set of vectors $v_1, v_2, \ldots, v_k$ all in V, and for every set of real numbers $a_1, a_2, \ldots, a_k$, the linear combination $a_1v_1 + a_2v_2 + \cdots + a_kv_k$ is a vector again in V.

   Closure 2: Both (a) "scalar multiplication" and (b) "vector addition" hold, where:
   (a) For every vector v in V, and every real number a, the product av is a vector again in V.
   (b) For every two vectors u, v in V, their sum $u + v$ is a vector again in V.

   Typically, if you already know that V is closed, you use Closure 1. However, if you want to *prove* that V is closed, you use Closure 2. There are other properties besides closure that must hold for V to be a vector space; we will study these in detail later.
2. Every vector space contains a zero vector. This could be it; we call this the "trivial vector space". If there is even one more vector, then there are infinitely many more; this can be proved by using scalar multiplication repeatedly.
3. If we set a linear function equal to a constant, e.g. $2x + 3y = 4$, we call this a *linear equation*.
4. The span is defined on (takes as input) a *set of vectors*, typically finite. Its product is (its value or output) is also a set of vectors. This product is an infinite set, with the sole exception of $Span(0) = \{0\}$.
5. "Spanning", "Dependent", and "Basis" are all properties that a *set of vectors* does or does not possess.
6. The standard basis for the linear function space on a set of variables, is exactly that set of variables. For example, the standard basis for the linear function space on $\{x, y\}$ is $\{x, y\}$.
7. The standard basis for $P_n(t)$ is the set $\{1, t, t^2, \ldots, t^n\}$. Note that this contains not n but $n+1$ vectors.
8. The standard basis for $P(t)$ is the set $\{1, t, t^2, \ldots\}$. Note that this contains infinitely many vectors.
9. The standard basis for $M_{m,n}$ is the set of mn matrices, each of which has all zero entries except for a single 1 entry. The mn possible locations of this 1 entry correspond to the different matrices. For example, $M_{2,2}$ has basis $\{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\}$. This is a set of $2 \times 2 = 4$ vectors.
10. The standard basis for $\mathbb{R}^n$ is denoted $\{e_1, e_2, \ldots, e_n\}$ where e_i has all zeroes, except for a single 1 in the i^{th} position. For example, if $n = 3$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.
11. If a set of vectors S contains two vectors, one of which is a multiple of the other, then S is dependent. For example, $S = \{1 + 2t, 3 + 5t, 2 + 4t\}$ is dependent because the first vector is half of the last one.
    WARNING: the reverse need not hold. A set of vectors could be dependent even if no vector is a multiple of another. For example, $T = \{1 + 2t, 3 + 5t, 4 + 7t\}$ is dependent because the sum of the first two vectors, minus the third, equals 0.
12. An important theorem we will learn later is that all bases of a vector space have the same size. This size is called the "dimension" of the vector space. Hence you now know the dimension of our most important vector spaces. For example, $P_2(t)$ is three dimensional; all of its bases consist of three vectors.
13. If a subset of a vector space is closed, that subset must itself be a vector space. We call this a subspace of the original vector space. This allows us to construct lots of new vector spaces, as subspaces of the important vector spaces you already know.

Helpful Proof Techniques:

1. Know your definitions, as 100% of all proofs (not just in this course, but in all of mathematics) rely heavily on the precise statements of definitions.
2. In particular, know the difference between a scalar (number), a vector, a set of vectors, and a vector

space. If you're working with something you need to *always* know which of these types it is.

3. To prove that a set of vectors S is closed, let u, v be arbitrary vectors in S, and a be an arbitrary real number. You need to prove that $u + v$ and au are both vectors in S.
4. To prove that a set of vectors S is not closed, you need a single counterexample. Either find some $u, v \in S$ where $u + v \notin S$, or find some $u \in S$ and $a \in \mathbb{R}$ where $au \notin S$. Sometimes only one of these two approaches will work.
5. To prove that a set of vectors S is spanning, take an arbitrary vector in V and show how to express it as a linear combination of S.
6. To prove that a set of vectors S is not spanning, you need a single counterexample. Select one vector in V (it may be hard to find one that works), assume that it can be expressed as a linear combination of S, and derive a contradiction.
7. To prove that a set of vectors S is dependent, you need to find a nondegenerate linear combination that gives the zero vector. This is typically harder the bigger S is.
8. To prove that a set of vectors S is independent, assume that a linear combination gives the zero vector, and prove that it must be the degenerate linear combination.
9. To prove that two sets are equal, prove that each is a subset of the other.

Solved Problems

1. Carefully state the definition of "Span".

   The span of a set of vectors $\{v_1, v_2, \ldots, v_k\}$ is the set of all linear combinations $\{a_1v_1 + a_2v_2 + \cdots + a_kv_k\}$, where the a_i each take on every real value.
2. Carefully state the definition of $P_3(t)$.

   $P_3(t)$ is the polynomial space in the variable t, of degree at most 3. Equivalently, this is $\{at^3 + bt^2 + ct + d\}$, where a, b, c, d each take on every real value.
3. Carefully state the definition of "Dependent".

   A set of vectors is dependent if their nondegenerate span contains the vector 0.
4. Carefully state the definition of $M_{2,2}$.

   $M_{2,2}$ is the matrix space consisting of all 2×2 matrices.
5. Carefully state the definition of "Basis".

   A basis is a set of vectors that is both spanning and independent.
6. Give two vectors from the linear function space in x.

   Many examples are possible, such as $3x, -4x, \pi x, 0$.
7. Give two vectors from $\mathbb{R}^4$.

   Many examples are possible, such as $(0, 0, 0, 1), (1, 2, 3, 4), (-1, 0, 0, 2)$.
8. Consider the vector space $\mathbb{R}^3$, and set $v = (-3, 2, 0), u = (0, 1, 4)$. Calculate $2v - u$.

   $2v - u = 2(-3, 2, 0) - (0, 1, 4) = (-6, 4, 0) + (0, -1, -4) = (-6, 3, -4)$
9. Consider the vector space $M_{2,3}$, and set $u = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$. Calculate $2v - u$.

   $2v - u = 2\begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 4 & -1 \\ -1 & 3 & 4 \end{pmatrix}$.
10. Consider the vector space $P(t)$, and set $u = t + 1, v = t + 2$. Prove that $3t + 1$ is in $Span(u, v)$.

    Note that $5u - 2v = 5(t+1) - 2(t+2) = 3t+1$, as desired. We find $5, -2$ by a side calculation; for example, $t = 2u - v$ and $1 = -u + v$ so $3t + 1 = 3(2u - v) + (-u + v) = 5u - 2v$. We will learn systematic ways to do this later.

11. Consider the vector space $P(t)$, and set $u = t + 1, v = t + 2$. Prove that $3t^2 + 1$ is not in $Span(u, v)$.

    Because u, v are both in $P_1(t)$, their span is as well (in fact it is exactly $P_1(t)$). However $3t^2 + 1$ is not in $P_1(t)$.

12. Consider the linear function space in $\{x, y, z\}$. Prove that $Span(x, y) = Span(x + y, x - y)$.

    Because $x + y = 1x + 1y$ and $x - y = 1x - 1y$, we conclude $x + y, x - y$ are each in $Span(x, y)$ and hence $Span(x + y, x - y) \subseteq Span(x, y)$. On the other hand, $x = \frac{1}{2}(x + y) + \frac{1}{2}(x - y)$ and $y = \frac{1}{2}(x + y) - \frac{1}{2}(x - y)$, so x, y are each in $Span(x + y, x - y)$ and hence $Span(x, y) \subseteq Span(x + y, x - y)$.

13. Consider the set S of all $v = (v_1, v_2)$ such that $|v_1| \geq |v_2|$. This is a subset of $\mathbb{R}^2$. Is it closed?

    For any scalar a and any vector v in S, we calculate $av = a(v_1, v_2) = (av_1, av_2)$. Because $|v_1| \geq |v_2|$, we may multiply both sides by the nonnegative $|a|$ to get $|a||v_1| \geq |a||v_2|$ and hence $|av_1| \geq |av_2|$. Hence av is a vector in S; the first closure property holds.
    We now take two vectors u, v in S, and calculate $u+v = (u_1, u_2)+(v_1, v_2) = (u_1+v_1, u_2+v_2)$. Must $|u_1 + v_1| \geq |u_2 + v_2|$? Perhaps not, so we need to find a specific counterexample. Many are possible, for example $u = (3, 1), v = (-3, 1)$. Both of u, v are in S, but $u + v = (0, 2)$ is not. Hence the second closure property does NOT hold. Since both closure properties do not hold, S is not closed.

14. Consider vector space V, and vectors v_1, v_2 in V. Set $S = Span(v_1, v_2)$. Prove that S is closed (and hence a subspace of V).

    Let u, w be arbitrary vectors from $Span(u, v)$. Then there are real numbers a_1, a_2, b_1, b_2 such that $u = a_1v_1 + a_2v_2$ and $w = b_1v_1 + b_2v_2$. We have $u + w = a_1v_1 + a_2v_2 + b_1v_1 + b_2v_2 = (a_1 + b_1)v_1 + (a_2 + b_2)v_2$, so $u + w$ is in S. This proves closure of vector addition. Let c be an arbitrary real number. Then $cu = c(a_1v_1 + a_2v_2) = (ca_1)v_1 + (ca_2)v_2$. Hence cu is in S. This proves closure of scalar multiplication.

    In fact, a similar proof works not just for two vectors, but for any number.

15. Consider the vector space $P_2(t)$, and set $S = \{a_0 + a_1t + a_2t^2 : a_0 + a_1 + a_2 = 0\}$, a subset. Prove that S is closed.

    Let u, v be arbitrary vectors in S. Then there are real numbers $a_0, a_1, a_2, b_0, b_1, b_2$ such that $u = a_0 + a_1t + a_2t^2$ and $v = b_0 + b_1t + b_2t^2$, and also $a_0 + a_1 + a_2 = 0 = b_0 + b_1 + b_2$. We have $u + v = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$, and $(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) = 0$, so $u + v$ is in S. This proves closure of vector addition. Let c be an arbitrary real number. Then $cu = (ca_0) + (ca_1)t + (ca_2)t^2$. We have $(ca_0) + (ca_1) + (ca_2) = c(a_0 + a_1 + a_2) = c0 = 0$, so cu is in S. This proves closure of scalar multiplication.

16. Consider the vector space $P(t)$, and set $u = t - 1, v = t^2 - 1, w = t^2 - t$. Prove that $3t + 1$ is not in $Span(u, v, w)$.

    Method 1: Suppose $3t + 1 = a(t - 1) + b(t^2 - 1) + c(t^2 - t) = (b + c)t^2 + (a - c)t - (a + b)$. Equating coefficients of the polynomials in t, we conclude that $b + c = 0, a - c = 3, -a - b = 1$. Adding these three equations we get $0 = 4$; hence there is no solution.
    Method 2: Let $S = \{a_0 + a_1t + a_2t^2 : a_0 + a_1 + a_2 = 0\}$, a subset of $P_2(t)$. S is closed by the preceding problem. Since $u, v, w \in S$, also $Span(u, v, w) \subseteq S$. However $3t + 1$ is not in S, so it cannot be in $Span(u, v, w)$.

17. Consider the vector space $\mathbb{R}^2$, and set $u = (1, 1), v = (2, 3), w = (0, 5)$. Prove that $\{u, v, w\}$ is dependent.

    To prove that $\{u, v, w\}$ is dependent, we need to find a nondegenerate linear combination yielding zero. Consider $10u - 5v + w$, found by a side calculation. $10u - 5v + w = 10(1, 1) - 5(2, 3) + (0, 5) = (10, 10) - (10, 15) + (0, 5) = (0, 0)$. Hence, $\{u, v, w\}$ is dependent.

18. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 2), v = (3, 0)$. Prove that $\{u, v\}$ is independent.

    To prove that $\{u, v\}$ is independent, we need to prove that any nondegenerate linear combination does not yield the zero vector. Suppose, to the contrary, that there *were* such a linear combination, i.e. some constants a, b (not both zero) so that $au + bv = (0, 0)$. We calculate $au + bv = a(2, 2) + b(3, 0) = (2a, 2a) + (3b, 0) = (2a + 3b, 2a) = (0, 0)$. So, we must have $2a + 3b = 0$ and $2a = 0$. The second equation gives us $a = 0$; we plug that into the first equation and get $b = 0$. Hence, $a = b = 0$ and the linear combination was actually degenerate (a contradiction!). Hence $\{u, v\}$ is independent.

19. Consider the vector space $\mathbb{R}^3$, and set $u = (1, 1, 1), v = (-1, 0, 1), w = (1, 2, 3)$. Prove that $\{u, v, w\}$ is dependent.

    To prove that $\{u, v, w\}$ is dependent requires a nondegenerate linear combination yielding the zero vector. We have $2u + v - w = 2(1, 1, 1) + 1(-1, 0, 1) - 1(1, 2, 3) = (2, 2, 2) + (-1, 0, 1) + (-1, -2, -3) = (0, 0, 0)$, so this set is dependent. To find this linear combination, we seek constants a, b, c (not all zero) so that $au + bv + cw = (0, 0, 0)$. We calculate $au + bv + cw = (a, a, a) + (-b, 0, b) + (c, 2c, 3c) = (a - b + c, a + 2c, a + b + 3c) = (0, 0, 0)$. Hence $a - b + c = 0, a + 2c = 0, a + b + 3c = 0$. This system has infinitely many solutions – choose c arbitrarily, then $a = -2c, b = -c$. The example above corresponded to $c = -1$.

    NOTE: No one of u, v, w is a multiple of any one of the others, and yet they are dependent.

20. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 3)$. Prove that $\{u\}$ is not spanning.

    To prove that $\{u\}$ is not spanning, we must provide a counterexample. We claim that $(1, 1)$ cannot be expressed as a linear combination of u, because then for some a we have $(1, 1) = a(2, 3) = (2a, 3a)$, and hence $2a = 1 = 3a$, which is impossible.

21. Consider the vector space $P_1(t)$. Prove that $\{t + 1, 2t - 1\}$ is spanning.

    Consider an arbitrary vector in $P_1(t)$, say $at + b$. We consider the linear combination $\alpha(t + 1) + \beta(2t - 1)$, where α, β are real numbers given by $\alpha = \frac{a+2b}{3}$ and $\beta = \frac{a-b}{3}$ (found by a side calculation). We compute that $\alpha(t + 1) + \beta(2t - 1) = \frac{a+2b}{3}(t + 1) + \frac{a-b}{3}(2t - 1) = t(\frac{a+2b}{3} + 2\frac{a-b}{3}) + (\frac{a+2b}{3} - \frac{a-b}{3}) = at + b$, as desired.

22. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 2), v = (3, 0)$. Prove that $\{u, v\}$ is spanning.

    To prove that $\{u, v\}$ is spanning, we need to prove that every vector can be expressed as a linear combination of u, v. Let $x = (x_1, x_2)$ be an arbitrary vector in $\mathbb{R}^2$. Set $a = x_2/2$ and set $b = (x_1 - x_2)/3$ (both real numbers no matter what x is), found by a side calculation. We have $au + bv = a(2, 2) + b(3, 0) = (2a + 3b, 2a) = (x_1, x_2) = x$.

23. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 2), v = (3, 0), w = (7, 5)$. Prove that $\{u, v, w\}$ is spanning.

    To prove that $\{u, v, w\}$ is spanning, we need to prove that every vector can be expressed as a linear combination of u, v, w. Comparing with the previous problem, already every $x = au + bv$, for some real a, b. Hence $x = au + bv + 0w$, a linear combination of $\{u, v, w\}$, so this set is also spanning.

24. Consider the vector space $\mathbb{R}^3$, and set $u = (1, 1, 1), v = (-1, 0, 1), w = (1, 2, 3)$. Prove that $\{u, v, w\}$ is not spanning.

    To prove that $\{u, v, w\}$ is not spanning, we must find a counterexample. We claim that $x = (1, 1, 0)$ is such a counterexample (found by a tricky side calculation). Suppose we could express x as a linear combination of u, v, w. Then, for some real constants a, b, c, we have $x = au + bv + cw = (a - b + c, a + 2c, a + b + 3c) = (1, 1, 0)$. Hence $a - b + c = 1, a + 2c = 1, a + b + 3c = 0$. Adding the first and third equations gives $2a + 4c = 1$, which is inconsistent with the second equation. Hence $x = (1, 1, 0)$ is not expressible as a linear combination of $\{u, v, w\}$, which is therefore not spanning.

25. Find two different bases for $\mathbb{R}^2$.

    Many solutions are possible. An easy choice is the standard basis $\{e_1, e_2\} = \{(1,0),(0,1)\}$. An earlier problem showed that $\{(2,2),(3,0)\}$ is spanning, and another proved that $\{(2,2),(3,0)\}$ is independent; hence this set is a basis.

26. Consider the linear function space in $\{x,y,z\}$. Set $S = Span(x+y, x+z)$. Find two bases for S.

    A natural choice is $\{x+y, x+z\}$; this set is spanning since $Span(x+y, x+z) = S$ is exactly what we need. This set is independent because if $a(x+y)+b(x+z)=0$ then $a=b=0$ so no nondegenerate linear combination gives 0.
    For another basis, consider $\{x+y, -y+z\}$. These are both vectors from S since $-y+z = -1(x+y)+1(x+z)$. This set is independent because if $a(x+y)+b(-y+z)=0$ then $a=b=0$ again. To prove it is spanning it is enough to prove $S \subseteq Span(x+y, -y+z)$. We have $x+y = 1(x+y)+0(-y+z)$, and $x+z = 1(x+y)+1(-y+z)$; hence the proof is complete.

Supplementary Problems

27. Carefully state the definition of "Vector Space", and give ten examples.
28. Carefully state the definition of "Span", and find a set of vectors whose span is itself.
29. Carefully state the definition of "Nondegenerate Span", and give two examples.
30. Carefully state the definition of "$M_{m,n}$", and give two vectors from $M_{3,2}$.
31. Carefully state the definition of "Independent", and give two examples from $P_2(t)$.
32. Consider the vectors in $\mathbb{R}^3$ given by $u = (1,2,3), v = (4,0,1), w = (-3,-2,5)$. Calculate $2u - 3v - 4w$.
33. Consider $S \subseteq \mathbb{R}^2$ of those vectors (v_1, v_2) such that $2v_1 + v_2 = 0$. Determine if S is closed.
34. Consider $S \subseteq \mathbb{R}^2$ of those vectors (v_1, v_2) such that $v_1 v_2 = 0$. Determine whether or not S is closed.
35. Consider the linear function space on $\{x,y,z\}$. Determine whether or not $x \in Span(x+y, x-z, y+z)$.
36. Consider the linear function space on $\{x,y,z\}$. Determine whether or not $x \in Span(x+y, x+z, y+z)$.
37. Consider the linear function space on $\{x,y,z\}$. Determine whether or not $x \in Span(x-y, x-z, y-z)$.
38. Consider $S \subseteq M_{2,2}$ of those vectors $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $c = 0$. Determine whether or not this is closed.
39. Consider $S \subseteq M_{2,2}$ of those vectors $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $a + c = 0$. Determine whether or not this is closed.
40. Consider $S \subseteq M_{2,2}$ of those vectors $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $a + c = 1$. Determine whether or not this is closed.
41. Consider the vector space $\mathbb{R}^2$, and set $u = (2,6), v = (-3,-9)$. Determine if $\{u, v\}$ is independent.
42. Consider $\mathbb{R}^2$, and set $u = (2,6), v = (-3,-9), w = (5,15)$. Determine if $\{u,v,w\}$ is independent.
43. Consider the vector space $\mathbb{R}^2$, and set $u = (2,6), v = (0,-9)$. Determine if $\{u, v\}$ is independent.
44. Consider the vector space $P_1(t)$. Determine whether or not $\{1, 2t\}$ is independent.
45. Consider the vector space $P_1(t)$. Determine whether or not $\{0, 1, 2t\}$ is spanning.
46. Consider the vector space $P_1(t)$. Determine whether or not $\{6t+2, -9t-3\}$ is spanning.
47. Consider the vector space $M_{2,2}$. Determine if $\left\{\left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\}$ is spanning.
48. Consider the vector space $M_{2,2}$. Determine if $\left\{\left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)\right\}$ is spanning.
49. Consider the vector space $M_{2,2}$. Determine if $\left\{\left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -1 \\ -1 & 0 \end{smallmatrix}\right)\right\}$ is spanning.
50. Which of the sets given in problems 41-49 are bases of their respective vector spaces?

Answers to Supplementary Problems: (WARNING: these are just answers, NOT thoroughly justified solutions)

32: $(2,12,-17)$ 33: yes 34: no 35: yes 36: yes 37: no 38: yes 39: yes 40: no 41: no 42: no 43: yes 44: yes 45: yes 46: no 47: no 48: no 49: yes 50: 43,44,49