Math 254 Fall 2014 Exam 10 Solutions

1. Carefully state the definition of "matrix space" $M_{m,n}$. Name every six-dimensional matrix space.

$M_{m,n}$ is the vector space consisting of all $m \times n$ matrices with real entries. There are four six-dimensional ones: $M_{1,6}, M_{2,3}, M_{3,2}, M_{6,1}$.

2. True or false: For all square matrices A, $det(A^2) \geq det(A)$. Be sure to justify your answer.

The statement is false. We need just one counterexample, and to help us find it we seek $det(A) > det(A^2) = det(A)^2$, which rearranges to $0 > det(A)(det(A) - 1)$, so we want a matrix A with $0 < det(A) < 1$. Examples of such are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0.5 \end{smallmatrix}\right)$ or just (0.5).

The remaining three problems concern the matrix $A = \begin{pmatrix} 1 & 2 & a \\ a & a & 1 \\ 1 & -1 & a \end{pmatrix}$, and the closely related system of linear equations $S = \{x + 2y + az = 1, ax + ay + z = 1, x - y + az = 1\}$.

3. Calculate $|A|$ using Laplace expansion on the first column. Be sure to simplify your answer.

We have $|A| = 1 \left|\begin{smallmatrix} a & 1 \\ -1 & a \end{smallmatrix}\right| - a \left|\begin{smallmatrix} 2 & a \\ -1 & a \end{smallmatrix}\right| + 1 \left|\begin{smallmatrix} 2 & a \\ a & 1 \end{smallmatrix}\right| = (a^2 + 1) - a(3a) + (2 - a^2) = 3 - 3a^2$.

4. Determine for which values of a (if any) the system S has a unique solution.

S will fail to have a unique solution exactly when $|A| = 0$. From (3), this is equivalent to $3 - 3a^2 = 0$, or $a^2 = 1$. Hence S will have a solution for all a except $a = 1, -1$.

5. Determine which values of a (if any) will lead to the system S having a unique solution where $x = 3$.

We use Cramer's rule: $3 = x = \frac{|A_1|}{|A|}$, or $3|A| = |A_1|$. We calculate $|A_1| = \begin{vmatrix} 1 & 2 & a \\ 1 & a & 1 \\ 1 & -1 & a \end{vmatrix} = 3 - 3a$, which we compute in any way we like. Hence $3(3 - 3a^2) = 3 - 3a$. Dividing both sides by 3 we get $3 - 3a^2 = 1 - a$, or $3a^2 - a - 2 = 0$. This has roots $a = -\frac{2}{3}, a = 1$; however $a = 1$ is invalid, without a unique solution. Hence $a = -\frac{2}{3}$ is the sole solution.