# Math 254 Fall 2014 Exam 11 Solutions

1. Carefully state the definition of "nondegenerate span". Give a set of vectors from $P_2(t)$ whose nondegenerate span does not include $t$.

   The nondegenerate span of a set of vectors is the set of all their linear combinations *except* the all-zero linear combination. The set $\{1, t^2\}$ meets the conditions.

2. Let $A \in M_{n,n}$. Suppose that $u, v$ are two eigenvectors of $A$, both corresponding to the same eigenvalue $\lambda$. Prove that $u + v$ is also an eigenvector for $A$ corresponding to $\lambda$.

   Because $u, v$ are eigenvectors, we have $Au = \lambda u$ and $Av = \lambda v$. Adding, we get $A(u + v) = Au + Av = \lambda u + \lambda v = \lambda(u + v)$. Hence $u + v$ is an eigenvector.

3. Consider $A = \begin{pmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$. Find all the eigenvalues of $A$, and for each give the algebraic and geometric multiplicities. Find the characteristic and minimal polynomials of $A$.

   $A$ is in JCF. Its only diagonal entry is 7, which appears 4 times. Hence $\lambda = 7$ is the only eigenvalue, with algebraic multiplicity 4. This means the characteristic polynomial is $\Delta_A(t) = (t - 7)^4$. There are two Jordan blocks, hence the geometric multiplicity of $\lambda = 7$ is 2. The largest Jordan block is size 3, hence the minimal polynomial is $\min_A(t) = (t - 7)^3$.

4. Suppose for matrix $B$ we have found the characteristic and minimal polynomials as $\Delta_B(t) = (t-3)^2(t-5)^2$, $\min_B(t) = (t-3)^2(t-5)$. Give the Jordan canonical form for $B$.

   $\lambda = 3$ is an eigenvalue of algebraic multiplicity 2 (from $\Delta_B(t)$). The largest Jordan block is of size 2 (from $\min_B(t)$), so that is the only Jordan block. $\lambda = 5$ is an eigenvalue of algebraic multiplicity 2 (from $\Delta_B(t)$). The largest Jordan block is of size 1 (from $\min_B(t)$), so there must be two such blocks. Thus the JCF of $B$ is $\begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

5. Find all the eigenvalues and eigenspaces of $C = \begin{pmatrix} 2 & -3 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$.

   We begin by calculating $\Delta_C(t) = det(tI - C) = \begin{vmatrix} t-2 & 3 & -4 \\ 0 & t+1 & -3 \\ 0 & 0 & t-2 \end{vmatrix} = (t-2)^2(t+1)$, which is easy to calculate since the matrix is triangular. Hence the eigenvalues are $\lambda = 2$ and $\lambda = -1$.

   The eigenspace corresponding to $\lambda = 2$ is the nullspace of $2I - C = \begin{pmatrix} 0 & 3 & -4 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$, which has row canonical form $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Hence this eigenspace has basis $\{(1, 0, 0)\}$.

   The eigenspace corresponding to $\lambda = -1$ is the nullspace of $2I - C = \begin{pmatrix} -3 & 3 & -4 \\ 0 & 0 & -3 \\ 0 & 0 & -3 \end{pmatrix}$, which has row canonical form $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Hence this eigenspace has basis $\{(1, 1, 0)\}$.