Math 254 Fall 2014 Exam 5 Solutions

1. Carefully state the definition of "vector space". (you need not list the axioms). Give two explicit examples.

   A vector space is a set of objects (called vectors), an underlying field (typically $\mathbb{R}$ for us), and some way to add vectors and multiply vectors by scalars, so long as eight axioms are upheld. Two explicit examples are $\mathbb{R}^2$ and $P_2(t)$.

2. Give the standard basis for $M_{3,2}$.

   The standard basis contains the following six vectors, not necessarily in this order: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

   The remaining three problems concern the vector space $V = \mathbb{R}^4$ and the subspaces $S = \{(a, b, c, d) : a + b = c + d = 0\}, \quad T = \{(a, b, c, d) : a + c = b + d = 0\}$.

3. Find a basis for $S \cap T$.

   We row reduce $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, in row canonical form. This has three pivots, and hence a one-dimensional solution space. A basis for this space is $\{(1, -1, -1, 1)\}$. Note: it is not enough to find a vector in $S \cap T$, you need to prove that the space is one-dimensional.

4. Find a basis for S, a basis for T, and a basis for $S + T$.

   We have $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ already in row canonical form; a basis for S is $\{(-1, 1, 0, 0), (0, 0, 1, -1)\}$.
   We have $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ already in row canonical form; a basis for T is $\{(-1, 0, 1, 0), (0, 1, 0, -1)\}$.
   We now row reduce $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, in row canonical form. This has three pivots; hence $S+T$ is three-dimensional, with basis $\{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1)\}$.

5. Use your above answers to find the dimensions of the four vector spaces $S, T, S \cap T, S + T$, and interpret these in terms of the Dimension Theorem.

   Counting the number of vectors in these bases, we get $dim(S) = dim(T) = 2, dim(S \cap T) = 1, dim(S + T) = 3$. The dimension theorem tells us that $dim(S) + dim(T) = dim(S \cap T) + dim(S + T)$, which indeed holds true here.