Math 254 Fall 2014 Exam 8 Solutions

1. **Carefully state the definition of "basis". Give a basis for the nullspace of $\left(\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 5 & 2 \end{smallmatrix}\right)$.**

   A basis in a vector space is a set of vectors that is independent and spanning. The nullspace of this matrix is a subspace of $\mathbb{R}^3$, one-dimensional since there is one free variable, so a basis is $\{(0, -2, 5)\}$

2. **Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and $F \circ F = I_2$ (identity). Prove that the nullity of F is 0, and find such an F.**

   If the nullity weren't zero, there would be a nonzero vector v such that $F(v) = 0$. But then $F \circ F(v) = F(F(v)) = F(0) = 0 \neq v$, which is a contradiction. Hence the nullity is zero. Many examples are possible, such as $F((a, b)) = (-a, -b), G((a, b)) = (b, a), H((a, b)) = (-a, b)$.

The remaining problems concern the function $F : P_2(t) \to \mathbb{R}^2$ given by $F(p(t)) = (p(2), p(-1))$.

3. **Prove that F is a linear transformation.**

   Let $p(t), q(t)$ be arbitrary polynomials in $P_2(t)$, and let k be an arbitrary real number. First, $F(p(t) + q(t)) = F((p + q)(t)) = ((p + q)(2), (p + q)(-1)) = (p(2) + q(2), p(-1) + q(-1)) = (p(2), p(-1)) + ((q(2), q(-1)) = F(p(t)) + F(q(t))$. Second, $F(kp(t)) = F((kp)(t)) = ((kp)(2), (kp)(-1)) = (k(p(2)), k(p(-1))) = k(p(2), p(-1)) = kF(p(t))$.

4. **Find the nullity of F, and a basis for its kernel.**

   Solution 1: Let $p(t) \in Ker(F)$. Write $p(t) = a + bt + ct^2$. We have $0 = F(p(t)) = (p(2), p(-1)) = (a + 2b + 4c, a - b + c)$. Hence $Ker(F)$ is isomorphic to the nullspace of matrix $\left(\begin{smallmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \end{smallmatrix}\right)$, which has row canonical form $\left(\begin{smallmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{smallmatrix}\right)$. This has one free variable, hence nullity(F)=1, and basis $\{(-2, -1, 1)\}$. Hence $Ker(F)$ has basis $\{-2 - t + t^2\}$.

   Solution 2: Let $p(t) \in Ker(F)$. We have $p(2) = 0, p(-1) = 0$, so $(t-2)$ and $(t+1)$ each divide $p(t)$. Hence $p(t) = (t - 2)(t + 1)q(t)$, for some polynomial $q(t)$. However since $p(t) \in P_2(t)$, and $(t - 2)(t + 1)$ already has degree 2, in fact $q(t)$ must be a constant polynomial. Thus $Ker(F) = \{k(t^2 - t - 2) : k \in \mathbb{R}\}$, a one-dimensional space with basis $\{t^2 - t - 2\}$. Since one-dimensional, the nullity of F is 1.

5. **Find the rank of F, and a basis for its image.**

   Solution 1: We apply the rank-nullity theorem to the previous problem. Since $dim(P_2(t)) = 3$, the rank of F must be 2. Since $\mathbb{R}^2$ has dimension 2, in fact F is onto, so a basis for its image is $\{(1, 0), (0, 1)\}$ (or any basis for $\mathbb{R}^2$).

   Solution 2: We will show $(1, 0)$ and $(0, 1)$ are each in the image of F. Since these form a basis for the codomain, this will prove that F is onto. Further, $\{(1, 0), (0, 1)\}$ are a basis for its image, and the rank is 2. We solve $\{p(2) = 1, p(-1) = 0\}$ to find $p(t) = \frac{1}{3} + \frac{1}{3}t$, and solve $\{q(2) = 0, q(-1) = 1\}$ to find $q(t) = \frac{2}{3} - \frac{1}{3}t$.