Math 254 Spring 2014 Exam 2b Solutions

1. Carefully state the definition of "vector space". Give two examples.

   A vector space is a set of objects (called vectors) together with a way to add vectors and a way to multiply vectors by real numbers. Many examples are possible, such as $M_{2,3}$, $P(t)$, and $\mathbb{R}^7$.

2. Give a nonhomogeneous system of two linear equations in two unknowns, such that the associated homogeneous system has infinitely many solutions. Find a basis for this homogeneous system.

   One possible nonhomogeneous system is $\{x + 2y = 3, 2x + 4y = 6\}$, whose associated homogeneous system is $\{x + 2y = 0, 2x + 4y = 0\}$. A basis for this homogeneous system is $\{(2, -1)\}$.

3. What is partial pivoting and why would you use it?

   Partial pivoting is a modification to Gaussian elimination, namely to always choose as pivot in each column the element of maximal absolute value. The benefit of partial pivoting is an increase in stability of the algorithm, leading to diminished roundoff error.

The remaining two problems both concern the matrix $A = \begin{bmatrix} 2&4&2&0&6 \\ 2&5&2&1&7 \\ 2&3&2&1&3 \\ 3&4&3&1&4 \end{bmatrix}$.

4. Place A in echelon form. Be sure to justify each step.

   $A \to \begin{bmatrix} 2&4&2&0&6 \\ 0&1&0&1&1 \\ 0&-1&0&1&-3 \\ 0&-2&0&1&-5 \end{bmatrix} \to \begin{bmatrix} 2&4&2&0&6 \\ 0&1&0&1&1 \\ 0&0&0&2&-2 \\ 0&0&0&3&-3 \end{bmatrix} \to \begin{bmatrix} 2&4&2&0&6 \\ 0&1&0&1&1 \\ 0&0&0&2&-2 \\ 0&0&0&0&0 \end{bmatrix}$ This last matrix is in echelon form.

   Step 1: $R_2 - R_1 \to R_2, R_3 - R_1 \to R_3, R_4 - 1.5R_1 \to R_4$.
   Step 2: $R_3 + R_2 \to R_3, R_4 + 2R_2 \to R_4$.
   Step 3: $R_4 - 1.5R_3 \to R_4$.

   What is important in this problem is (1) Does the solution use correct, clear, and justified elementary operations, and (2) Is the final result in echelon form?

5. Place A in row canonical form. Be sure to justify each step. You should begin with your answer from (4).

   $\begin{bmatrix} 2&4&2&0&6 \\ 0&1&0&1&1 \\ 0&0&0&2&-2 \\ 0&0&0&0&0 \end{bmatrix} \to \begin{bmatrix} 1&2&1&0&3 \\ 0&1&0&1&1 \\ 0&0&0&1&-1 \\ 0&0&0&0&0 \end{bmatrix} \to \begin{bmatrix} 1&2&1&0&3 \\ 0&1&0&0&2 \\ 0&0&0&1&-1 \\ 0&0&0&0&0 \end{bmatrix} \to \begin{bmatrix} 1&0&1&0&-1 \\ 0&1&0&0&2 \\ 0&0&0&1&-1 \\ 0&0&0&0&0 \end{bmatrix}$ This last matrix is in row canonical form.

   Step 1: $0.5R_1 \to R_1, 0.5R_3 \to R_3$.
   Step 2: $R_2 - R_3 \to R_2$.
   Step 3: $R_1 - 2R_2 \to R_1$.

   What is important in this problem is (1) Does the solution use correct, clear, and justified elementary operations, and (2) Is the final result in row canonical form?