Math 254 Spring 2014 Exam 3 Solutions

1. Carefully state the definition of "nondegenerate span". Give two sets of vectors from $P_1(t)$: one set called A whose nondegenerate span includes 0, and one set called B whose nondegenerate span includes everything except 0.

The nondegenerate span of a set of vectors is the set of all linear combinations of that set, *except* the degenerate (all-zero) linear combination. $A = \{t, 2t\}$, $B = \{1, t\}$ are examples of A, B; there are others.

2. Prove that: if A, B are each orthogonal matrices, then AB is also an orthogonal matrix.

Recall that matrix M is orthogonal if $MM^T = I$. We compute $(AB)(AB)^T = (AB)B^TA^T = A(BB^T)A^T = AIA^T = AA^T = I$. Hence AB is orthogonal.

The remaining three problems all concern the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$.

3. Find matrices B, C such that B is skew-symmetric, C is upper triangular, and $A = B + C$.

Since C is zero in the lower left corner, we must have $3, 2, 0$ in the lower left corner of B. Hence $B = \begin{bmatrix} 0 & -3 & -2 \\ 3 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, and calculate $C = A - B = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4. Determine whether A^{-1} exists; if yes, find it.

$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -3 & 1 & 0 \\ 0 & -2 & -2 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & -2 & 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1.5 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$
$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 & 1 & -1.5 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$. Hence A^{-1} exists and $A^{-1} = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 1 & -1.5 \\ 1 & -1 & 1 \end{bmatrix}$.

5. Compute $A^2 - 2A + I$.

Method 1: $A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 1 \\ 6 & 4 & 3 \\ 2 & 2 & 2 \end{bmatrix}$. $-2A = \begin{bmatrix} -2 & -2 & -2 \\ -6 & -2 & 0 \\ -4 & 0 & 0 \end{bmatrix}$. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Adding these three matrices gives $A^2 - 2A + I = \begin{bmatrix} 5 & 0 & -1 \\ 0 & 3 & 3 \\ -2 & 2 & 3 \end{bmatrix}$.

Method 2: $A^2 - 2A + I = (A - I)^2$. We calculate $A - I = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}$, then square $(A - I)^2 = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 3 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -1 \\ 0 & 3 & 3 \\ -2 & 2 & 3 \end{bmatrix}$.