# Math 254 Spring 2014 Exam 7 Solutions

1. Carefully state the definition of "basis". Give two bases for $P_2(t)$.

   A basis is a set of vectors that is linearly independent and spanning. Examples include $\{1, t, t^2\}, \{1, 2t, 3t^3\}, \{1, 1+t, 1+t+t^2\}$.

2. Let $u, v$ be nonzero vectors in an arbitrary inner product space. Prove that if $u$ is orthogonal to $v$ then $proj(u, v) = \overline{0}$.

   If $u$ is orthogonal to $v$ then $\langle u, v\rangle = 0$. We have $proj(u, v) = \frac{\langle u,v\rangle}{\langle v,v\rangle} v = \frac{0}{\langle v,v\rangle} v = 0v = \overline{0}$.

The remaining three problems are all in vector space $V = \mathbb{R}^2$, with inner product $\langle \cdot, \cdot\rangle_\star$ given by $\langle u, v\rangle_\star = u^T \left(\begin{smallmatrix} 4 & 1 \\ 1 & 3 \end{smallmatrix}\right) v$. (here $u, v$ are column vectors).

3. Let $a = (1, 1)^T$ and $b = (-1, 3)^T$. Calculate $\langle a, a\rangle_\star, \langle a, b\rangle_\star, \langle b, b\rangle_\star$.

   We have $\langle a, a\rangle_\star = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 1 \\ 1 & 3 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = 9$, $\langle a, b\rangle_\star = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 1 \\ 1 & 3 \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right) = 7$,
   $\langle b, b\rangle_\star = \left(\begin{smallmatrix} -1 & 3 \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 1 \\ 1 & 3 \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right) = 25$.

4. Let $a = (1, 1)^T$ and $b = (-1, 3)^T$. Find vectors $c, d$ such that $a = c + d$, $c$ is a multiple of $b$, and $d$ is orthogonal to $b$ (in the $\langle \cdot, \cdot\rangle_\star$ sense).

   We take $c = Proj(a, b) = \frac{\langle a,b\rangle_\star}{\langle b,b\rangle_\star} b = \frac{7}{25} b = (\frac{-7}{25}, \frac{21}{25})^T$. We now calculate $d = a - c = (1, 1)^T - (\frac{-7}{25}, \frac{21}{25})^T = (\frac{32}{25}, \frac{4}{25})^T$.
   It's not necessary, but we can check that $\langle d, b\rangle_\star = \left(\begin{smallmatrix} \frac{32}{25} & \frac{4}{25} \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & 1 \\ 1 & 3 \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right) = 0$.

5. Starting with $\{(1, 0)^T, (0, 1)^T\}$, use Gram-Schmidt to find an orthonormal basis for $V$ (in the $\langle \cdot, \cdot\rangle_\star$ sense).

   We have $f_1 = e_1$, and compute $f_2 = e_2 - Proj(e_2, e_1) = e_2 - \frac{\langle e_2,e_1\rangle_\star}{\langle e_1,e_1\rangle_\star} e_1 = e_2 - \frac{1}{4} e_1 = (-\frac{1}{4}, 1)$; hence an orthogonal basis is $\{f_1, f_2\} = \{(1, 0), (-\frac{1}{4}, 1)\}$. To normalize we need $\|f_1\|_\star = \sqrt{\langle f_1, f_1\rangle_\star} = \sqrt{4} = 2$ and $\|f_2\|_\star = \sqrt{\langle f_2, f_2\rangle_\star} = \sqrt{\frac{11}{4}} = \frac{\sqrt{11}}{2}$. Combining, we get orthonormal basis $\{\frac{f_1}{\|f_1\|_\star}, \frac{f_2}{\|f_2\|_\star}\} = \{(\frac{1}{2}, 0), (-\frac{1}{2\sqrt{11}}, \frac{2}{\sqrt{11}})\}$.