MATH 521A: Abstract Algebra
Homework 10 Solutions

1. Prove that $(6, 15, 27) = (3)$ in $\mathbb{Z}$.

   We have $(6, 15, 27) \subseteq (3)$ since $6 = 2 \cdot 3$, $15 = 5 \cdot 3$, and $27 = 9 \cdot 3$. We also have $(6, 15, 27) \supseteq (3)$ because $12 = 2 \cdot 6 \in (6, 15, 27)$ and hence $3 = 15 - 12 \in (6, 15, 27)$.

2. Find all ideals of $\mathbb{Z}_{12}$. Determine which of these are principal, maximal, and prime.

   The principal ideals are $(0) = \{0\}$, $(1) = (5) = (7) = (11) = \mathbb{Z}_{12}$, $(2) = (10) = \{0, 2, 4, 6, 8, 10\}$, $(3) = (9) = \{0, 3, 6, 9\}$, $(4) = (8) = \{0, 4, 8\}$, and $(6) = \{0, 6\}$. There are no nonprincipal ideals. [Proof: compose $\phi : \mathbb{Z} \to \mathbb{Z}_{12}$, $\pi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}/I$ to get a ring homomorphism, whose kernel is an ideal in $\mathbb{Z}$, which is principal since $\mathbb{Z}$ is a PID.] Of these, (2) and (3) are both maximal and prime.

3. Suppose I, J are ideals of some ring R. Prove that $I \cap J$ and $I + J$ are both ideals of R.

   $I \cap J$: Let $x, y \in I \cap J$. The $x + y, xy, -x$ are all in both I, J, so in $I \cap J$. Also $0 \in I \cap J$, so $I \cap J$ is a subring. Let $r \in R$. We have xr in both I, J, so in $I \cap J$. Hence $(I \cap J)R \subseteq (I \cap J)$.
   $I + J$: Let $a + b, a' + b' \in I + J$, for some $a, a' \in I, b, b' \in J$. $(a + b) + (a' + b') = (a + a') + (b + b') \in I + J$. $(a + b)(a' + b') = a(a' + b') + b(a' + b') \in I + J$ (since I, J ideals). $-(a + b) = (-a) + (-b) \in I + J$. $0 = 0 + 0 \in I + J$. Hence $I + J$ is a subring. Let $r \in R$. We have $(a + b)r = ar + br \in I + J$. Hence $I + J$ is an ideal.

4. Let R be a field. Prove that its only ideals are (0) and R.

   Let I be an ideal. If I contains no nonzero element, then $I = (0)$. Otherwise, let $x \in I$ be nonzero. Let $y \in R$ be arbitrary. $x^{-1}y \in R$, so $x(x^{-1}y) = y \in I$. Hence $I = R$.

5. Let R be a ring, and $a \in R$. Set $I = \{b \in R : ab = 0\}$. Prove that I is an ideal of R.

   Let $b, b' \in I$. We have $a(b + b') = ab + ab' = 0 + 0 = 0$, so $b + b' \in I$. We have $a(bb') = (ab)b' = 0$, so $bb' \in I$. We have $a(-b) = -(ab) = 0$, so $-b \in I$. We have $a0 = 0$, so $0 \in I$. Hence I is a subring of R. Let $r \in R$. We have $a(br) = (ab)r = 0$, so $br \in I$. Hence I is an ideal.

6. Calculate simple forms for the elements of the ideal $I = (6x, 10)$ in $R = \mathbb{Z}[x]$. Is it principal? Maximal? Prime?

   Note that $2x = 2 \cdot 6x - x \cdot 10 \in I$. Hence $I \subseteq (2x, 10)$. But also $6x = 3 \cdot 2x$, so in fact $I = (2x, 10)$. Thus a simple form for $I = \{10a_0 + 2a_1x + 2a_2x^2 + \cdots + 2a_nx^n : a_i \in \mathbb{Z}\}$. We calculate $R/I = \{b_0 + b_1x + \cdots + b_nx^n + I : b_0 \in [0, 9], b_i \in \{0, 1\}\}$. This is not an integral domain, as $2 + I, 5 + I \in R/I$ yet $(2 + I)(5 + I) = 0 + I$. Hence I is neither maximal nor prime. It is also not principal; to contain 10 it would have to be (a) for some integer a. The only such principal ideals containing $2x$ are $a \in \{2, -1, 1, 2\}$, and all of these are bigger than I.

7. Calculate simple forms for the elements of the ideal $I = (6x, 10x)$ in $R = \mathbb{Z}[x]$. Is it principal? Maximal? Prime?

   Note that $2x = 2 \cdot 6x - 1 \cdot 10x$, so $(2x) \subseteq I$. But also $6x = 3 \cdot 2x$ and $10x = 5 \cdot 2x$, so in fact $I = (2x)$. Thus I is principal. Now, $2x \in I$, but $2 \notin I$ and $x \notin I$. Hence I is not prime, and thus not maximal.

8. Prove that $\mathbb{Z}/20\mathbb{Z} \cong \mathbb{Z}_{20}$. Some people prefer to write $\mathbb{Z}/20\mathbb{Z}$ instead of $\mathbb{Z}_{20}$.

   Consider the surjective ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_{20}$ given by $\phi(x) = [x]_{20}$. The kernel is $\{x \in \mathbb{Z} : [x] = [0]\} = 20\mathbb{Z}$. We are now done by the First Isomorphism Theorem.

9. Let I, K be ideals in R, with $K \subseteq I$. Prove that $I/K = \{x + K : x \in I\}$ is an ideal in $R/K = \{x + K : x \in R\}$.

   I/K is a ring, contained in the ring R/K; hence it is a subring. Now, let $a + K \in I/K$ and $r + K \in R/K$. We have $(a + K)(r + K) = ar + aK + rK + KK$. Since K is an ideal, $aK + rK + KK \subseteq K$, so $(a + K)(r + K) = ar + K$. Since I is an ideal, $ar = b$ for some $b \in I$. Hence $(a + K)(r + K) = b + K \in I/K$.