MATH 521A: Abstract Algebra
Homework 5 Solutions

1. Let R be a ring with operations $\oplus, \odot$. Define its *annihilation ring* R^{ann} as follows. R^{ann} has the same ground set as R. We define addition in R^{ann} to be the same as in R, i.e. $\forall a, b \in R^{ann}$, $a \oplus^{ann} b = a \oplus b$. We define multiplication in R^{ann} as $\forall a, b \in R^{ann}$, $a \odot^{ann} b = 0_R$. Prove that R^{ann} is a ring.

Most of the ring axioms don't involve multiplication, so R^{ann} inherits them from R, since it has the same addition. Let $a, b, c \in R^{ann}$ be arbitrary. We have $a \odot^{ann} (b \odot^{ann} c) = a \odot^{ann} 0_R = 0_R = 0_R \odot^{ann} c = (a \odot^{ann} b) \odot^{ann} c$. We also have $a \odot^{ann} (b \oplus^{ann} c) = 0_R = 0_R \oplus 0_R = 0_R \oplus^{ann} 0_R = (a \odot^{ann} b) \oplus^{ann} (a \odot^{ann} c)$. Lastly, we have $(b \oplus^{ann} c) \odot^{ann} a = 0_R = 0_R \oplus^{ann} 0_R = (b \odot^{ann} a) \oplus^{ann} (c \odot^{ann} a)$.

2. Let R be a ring with just two elements: $\{0, a\}$. How many such rings are there? Be sure to prove your answer.

The addition table must be $0 + 0 = 0 = a + a$, and $0 + a = a + 0 = a$, because 0 is neutral and a must have an inverse. The multiplication table must have $0 \cdot 0 = 0 \cdot a = a \cdot 0 = 0$, by theorem 3.5. However we don't know if $a \cdot a = a$ or $a \cdot a = 0$. It turns out both are possible; the former is isomorphic to $\mathbb{Z}_2$, while the latter is isomorphic to $\mathbb{Z}_2^{ann}$.

3. Let R be a ring with identity with just three elements: $\{0, 1, a\}$. How many such rings are there? Be sure to prove your answer.

Consider first $1+a$. It can't equal 1, else $a = 0$. It can't equal a, else $1 = 0$. Hence $1+a = 0$. Now consider $1 + 1$. It can't equal 1, else $1 = 0$. It can't equal 0, else $1 + 1 = 0 = 1 + a$, so $a = 1$. Hence $1 + 1 = a$. Lastly, $a + a$ can't be 0, else $a + a = 0 = 1 + a$ so $a = 1$, and $a + a$ can't be a, else $a = 0$. Hence $a + a = 1$. Putting this all together gives the same addition table as $\mathbb{Z}_3$.

For the multiplication, we know that $0 = 0 \cdot 0 = 0 \cdot 0 = 0 \cdot 1 = 0 \cdot a = 1 \cdot 0 = a \cdot 0$. We also know that $1 \cdot 1 = 1$ and $1 \cdot a = a \cdot 1 = a$. The only mystery is $a \cdot a$. However we know that $a = 1 + 1$, so we have $a \cdot a = (1 + 1) \cdot (1 + 1) = 1 + 1 + 1 + 1 = a + a = 1$. Hence the multiplication agrees with $\mathbb{Z}_3$; so any such ring must be isomorphic to $\mathbb{Z}_3$.

4. Let R be a ring with identity. Suppose that $a, b \in R$ such that a, ab are both units. Prove that b is a unit. Do not assume that R is commutative.

Since a is a unit, there is some $c \in R$ with $ca = ac = 1_R$. Similarly, since ab is a unit, there is some $d \in R$ with $d(ab) = (ab)d = 1_R$. We will prove that $u = da$ is the reciprocal of b. First, multiply $1_R = abd$ on the left by c, to get $c = c1_R = (ca)bd = 1_R bd = bd$. Multiply this by a on the right to get $1_R = ca = b(da) = bu$. The other direction is easier; $ub = (da)b = d(ab) = 1_R$. Hence b is a unit.

5. Let $R = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{Q}\}$, the ring of 2×2 matrices over $\mathbb{Q}$, with operations of the usual matrix addition and matrix multiplication. Prove that every nonzero element of R is either a unit or a zero divisor.

The trick is to find a test that classifies elements, namely the determinant $ad-bc$. Claim 1: If $ad-bc \neq 0$, then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a unit. Proof: Set $f = ad-bc$ and just compute $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \begin{pmatrix} d/f & -b/f \\ -c/f & a/f \end{pmatrix} =$

$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)=1_R$.
Claim 2: If $ad-bc=0$, then $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is a (two-sided) zero divisor. Compute $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}d&-b\\-c&a\end{smallmatrix}\right)=\left(\begin{smallmatrix}d&-b\\-c&a\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\left(\begin{smallmatrix}ad-bc&0\\0&ad-bc\end{smallmatrix}\right)=\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)=0_R$.

6. Let R be a ring. Consider the *diagonal map* $f:R\to R\times R$ given by $f:r\mapsto(r,r)$. Prove that f is a (ring) homomorphism.

   Let $a,b\in R$. We have $f(a+b)=(a+b,a+b)=(a,a)+(b,b)=f(a)+f(b)$, and $f(ab)=(ab,ab)=(a,a)(b,b)=f(a)f(b)$.

7. Let R,S,T be rings. Prove that the ring $(R\times S)\times T$ is isomorphic to the ring $R\times(S\times T)$.

   We need to find a candidate isomorphism, and the natural choice is $f:((x,y),z)\mapsto(x,(y,z))$. First, let's prove it's a homomorphism. Let $x,x'\in R, y,y'\in S, z,z'\in T$, and we have $f(((x,y),z)+((x',y'),z'))=f(((x,y)+(x',y'),z+z'))=f(((x+x',y+y'),z+z'))=(x+x',(y+y',z+z'))=(x,(y,z))+(x',(y',z'))=f(((x,y),z))+f(((x',y'),z'))$. Similarly, we have $f(((x,y),z)((x',y'),z'))=f(((xx',yy'),zz'))=(xx',(yy',zz'))=(x,(y,z))(x',(y',z'))=f(((x,y),z)))f(((x',y'),z')))$.

   Now, to prove bijection, we need to prove surjectivity and injectivity. Suppose that $f(((x,y),z)))=f(((x',y'),z')))$. Then $(x,(y,z))=(x',(y',z'))$ and hence $x=x', y=y', z=z'$, so $((x,y),z)=((x',y'),z')$. This proves one-to-one. Lastly, let $(a,(b,c))\in R\times(S\times T)$. We see that $((a,b),c)\in(R\times S)\times T$ and $f(((a,b),c))=(a,(b,c))$. This proves onto.

8. Prove that $\mathbb{Z}_9$ is not isomorphic to $\mathbb{Z}_3\times\mathbb{Z}_3$, despite having the same number of elements.

   Suppose there were some isomorphism $f:\mathbb{Z}_9\to\mathbb{Z}_3\times\mathbb{Z}_3$. Note that if $a,b\in\mathbb{Z}_9$ with $ab=1_9$, then $f(a)f(b)=f(ab)=f(1_9)=1_{3\times3}$, so every unit in $\mathbb{Z}_9$ must map to a unit in $\mathbb{Z}_3\times\mathbb{Z}_3$. However we found six units in $\mathbb{Z}_9$ and only four in $\mathbb{Z}_3\times\mathbb{Z}_3$. Thus no such isomorphism can exist. One could consider zero divisors instead.

9. Consider the function $f:\mathbb{Z}_7\to\mathbb{Z}_{56}$ given by $f:[x]_7\mapsto[8x]_{56}$. Prove that f is an injective homomorphism, but not an isomorphism.

   Let $[x],[y]\in\mathbb{Z}_7$. We have $f([x]+[y])=f([x+y])=[8(x+y)]_{56}=[8x]_{56}+[8y]_{56}=f([x])+f([y])$, and $f([x][y])=f([xy])=[8(xy)]_{56}=[8(xy)]_{56}+[56(xy)]_{56}=[64(xy)]_{56}=[8x]_{56}[8y]_{56}=f([x])f([y])$. Hence f is a homomorphism. Proving injectivity is as simple as noting the image of f is $\{[0]_{56},[8]_{56},[16]_{56},[24]_{56},[32]_{56},[40]_{56},[48]_{56}\}$, which has just seven elements. Since $7<56$, f is not surjective.

10. Consider the ring R, on ground set $\mathbb{Z}$, with operations $\oplus,\odot$ defined as $a\oplus b=a+b+1$, $a\odot b=ab+a+b$. Prove that R is isomorphic to $\mathbb{Z}$. (you may assume that R is a ring)

    The hard part is finding the right isomorphism, which is $f:R\to\mathbb{Z}$ given by $f(x)=x+1$. First, let's prove homomorphism. Let $a,b\in\mathbb{Z}$. We have $f(a\oplus b)=f(a+b+1)=a+b+2=(a+1)+(b+1)=f(a)+f(b)$. We also have $f(a\odot b)=f(ab+a+b)=ab+a+b+1=(a+1)(b+1)=f(a)f(b)$. Lastly we prove isomorphism. Suppose that $f(a)=f(b)$. Then $a+1=b+1$, so $a=b$. Hence f is injective. Let $a\in\mathbb{Z}$; we have $f(a-1)=(a-1)+1=a$, so f is surjective.