MATH 521A: Abstract Algebra
Homework 5: Due Oct.11

1. Let R be a ring with operations $\oplus, \odot$. Define its *annihilation ring* R^{ann} as follows. R^{ann} has the same ground set as R. We define addition in R^{ann} to be the same as in R, i.e. $\forall a, b \in R^{ann}$, $a \oplus^{ann} b = a \oplus b$. We define multiplication in R^{ann} as $\forall a, b \in R^{ann}$, $a \odot^{ann} b = 0_R$. Prove that R^{ann} is a ring.

2. Let R be a ring with just two elements: $\{0, a\}$. How many such rings are there? Be sure to prove your answer.

3. * Let R be a ring with identity with just three elements: $\{0, 1, a\}$. How many such rings are there? Be sure to prove your answer.

4. Let R be a ring with identity. Suppose that $a, b \in R$ such that a, ab are both units. Prove that b is a unit. Do not assume that R is commutative.

5. Let $R = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{Q}\}$, the ring of 2×2 matrices over $\mathbb{Q}$, with operations of the usual matrix addition and matrix multiplication. Prove that every nonzero element of R is either a unit or a zero divisor.

6. Let R be a ring. Consider the *diagonal map* $f : R \to R \times R$ given by $f : r \mapsto (r, r)$. Prove that f is a (ring) homomorphism.

7. * Let R, S, T be rings. Prove that the ring $(R \times S) \times T$ is isomorphic to the ring $R \times (S \times T)$.

8. Prove that $\mathbb{Z}_9$ is not isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, despite having the same number of elements.

9. Consider the function $f : \mathbb{Z}_7 \to \mathbb{Z}_{56}$ given by $f : [x]_7 \mapsto [8x]_{56}$. Prove that f is an injective homomorphism, but not an isomorphism.

10. Consider the ring R, on ground set $\mathbb{Z}$, with operations $\oplus, \odot$ defined as $a \oplus b = a + b + 1$, $a \odot b = ab + a + b$. Prove that R is isomorphic to $\mathbb{Z}$. (you may assume that R is a ring)