MATH 521A: Abstract Algebra

Homework 8: Due Nov. 8

1. * For nonzero polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$, define the *content* of $f(x)$ as $c(f) = \gcd(a_n, a_{n-1}, \ldots, a_1, a_0)$. We call f *primitive* if $c(f) = 1$. Let $f(x), g(x) \in \mathbb{Z}[x]$. Suppose that $f(x), g(x)$ are both primitive. Prove that their product $f(x)g(x)$ is also primitive.

2. For nonzero $f(x), g(x) \in \mathbb{Z}[x]$, prove that $c(fg) = c(f)c(g)$.

3. * Let $f(x) \in \mathbb{Z}[x]$. Suppose that there are non-units $g(x), h(x) \in \mathbb{Q}[x]$ such that $f(x) = g(x)h(x)$. Then there are $g'(x), h'(x) \in \mathbb{Z}[x]$ such that $f(x) = g'(x)h'(x)$ and $\deg g(x) = \deg g'(x)$ (and also $\deg h(x) = \deg h'(x)$).
   Note: $g'(x)$ is just another polynomial, not a derivative.

4. Fix $a \in \mathbb{Z}$ and consider $\phi_a : \mathbb{Z}[x] \to \mathbb{Z}[x]$ given by $\phi_a : f(x) \mapsto f(x - a)$. Prove that if $f(x)$ is reducible then $\phi_a(f(x))$ is reducible.

5. Use Eisenstein's criterion (and Problem 4, if necessary) to prove that $x^5 + 5x + 2$ is irreducible in $\mathbb{Q}[x]$.

6. Fix p prime, and consider the "natural map" $\phi_p : \mathbb{Z}[x] \to \mathbb{Z}_p[x]$ given by $\phi_p : a_n x^n + \cdots + a_1 x + a_0 \mapsto [a_n]_p x^n + \cdots + [a_1]_p x + [a_0]_p$. Prove that if $p \nmid a_n$ and $f(x)$ is primitive and reducible, then $\phi_p(f(x))$ is also reducible.

7. Use Problem 6 to prove that $f(x) = x^3 + 5x + 4$ is irreducible in $\mathbb{Z}[x]$.

8. Set $f(x) = 3x^3 + 4x^2 + 7x + 2$. Show that this is reducible in $\mathbb{Z}[x]$ but irreducible in $\mathbb{Z}_3[x]$. Does this contradict problem 6?

9. Factor $x^4 - 25$ in $\mathbb{Q}[x]$, $\mathbb{R}[x]$, and $\mathbb{C}[x]$.

10. Factor $x^3 - ix^2 + 5x - 5i$ in $\mathbb{C}[x]$.