**MATH 521B: Abstract Algebra**
Preparation for Exam 2

Please recall the following:
$\mathbb{C}^\times = \{x \in \mathbb{C} : x \neq 0\}$ and $\mathbb{R}^\times = \{x \in \mathbb{R} : x \neq 0\}$ are groups under multiplication. $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ are groups under addition. $S_n$ is the symmetric group on $[n]$, and $A_n \leq S_n$ is the alternating group on $[n]$ that consists of even permutations. $A \times B$ denotes the external direct product $\{(a,b) : a \in A, b \in B\}$. If $A, B$ are groups then so is $A \times B$ with group operation $(a,b)(a',b') = (aa', bb')$.

1. Set $G = \{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Calculate the Cayley table for $S_4/G$.
2. Set $G = \{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Prove that $G \trianglelefteq A_4$, and calculate the Cayley table for $A_4/G$.
3. Fix $H = \langle (1\ 2\ 3) \rangle \leq S_3$. Prove that $H \trianglelefteq S_3$, and find a familiar group isomorphic to $S_3/H$.

   For problems 4-7, let $G = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a, b, d \in \mathbb{R}, ad \neq 0 \right\}$, and $N = \{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{R}\}$.
4. Prove that $N \trianglelefteq G$.
5. For each $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = x \in G$, determine explicitly all $y \in G$ with $x \equiv y \pmod{N}$.
6. Prove that $N \cong \mathbb{R}$.
7. Prove that $G/N \cong \mathbb{R}^\times \times \mathbb{R}^\times$.

   For problems 8-9, define $U \leq \mathbb{C}^\times$ via $U = \{a + bi : a^2 + b^2 = 1\}$.
8. Prove that $\mathbb{C}^\times / U \cong \mathbb{R}^\times$.
9. Prove that $\mathbb{R}/\mathbb{Z} \cong U$.
10. Fix a finite group $G$, with $N \trianglelefteq G$. Set $m = [G : N]$. Prove that $a^m \in N$, for all $a \in G$.
11. Fix abelian group $G$, with $|G| = 2k$, and $k$ odd. Prove that $G$ has exactly one element $g$ with $|g| = 2$.
12. Let $p$ be an odd prime, and let $G$ be a nonabelian group with $|G| = 2p$. Prove that $G$ contains an element of order $p$.
13. Fix abelian group $G$. Set $K = \{g \in G : |g| \leq 2\}$, $H = \{x^2 : x \in G\}$. Prove that $K \trianglelefteq G$, $H \trianglelefteq G$, and that $G/K \cong H$.
14. Let $f : G \to H$ be an onto homomorphism. Suppose $N \trianglelefteq G$. Prove that $f(N) \trianglelefteq H$.
15. Fix groups $G, H$, and suppose $M \trianglelefteq G$ and $N \trianglelefteq H$. Prove that $(M \times N) \trianglelefteq (G \times H)$.
16. Fix groups $G, H$, and suppose $M \trianglelefteq G$ and $N \trianglelefteq H$. Prove that $(G/M) \times (H/N) \cong (G \times H)/(M \times N)$.
17. Fix a finite group $G$, and some $s \in \mathbb{N}$. Set $T = \{K : |K| = s,\ K \leq G\}$, the set of all subgroups of order $s$, and assume $|T| \geq 1$. Set $N = \bigcap T$. Prove that $N \trianglelefteq G$.
18. Fix $n \geq 5$. Set $T = \{(r\ s\ t)\} \subseteq S_n$, the set of all permutations that consist of a single cycle of length three. Prove that $A_n = \langle T \rangle$.
19. Fix $n \geq 5$ and $N \trianglelefteq A_n$. Set $T = \{(r\ s\ t)\} \subseteq S_n$, the set of all permutations consisting of just a cycle of length three. If $N \cap T \neq \emptyset$, prove that $N = A_n$.
20. Fix a group $G$, and suppose $N \trianglelefteq G$. Suppose $N$ is maximal normal, i.e. there is no $M$ with $N < M \triangleleft G$. Prove that $G/N$ is simple.