MATH 521B: Abstract Algebra
Preparation for Exam 3

1. For $G = \mathbb{Z}_2 \oplus \mathbb{Z}_6$, determine $G(2)$ and $G(3)$ explicitly.

2. I've thought of an integer between 1 and 1000. Dividing by 7 gives a remainder of 1, dividing by 11 gives a remainder of 6, and dividing by 13 gives a remainder of 3. What is my number?

3. Let G, H be finite abelian groups of the same order n. Prove that $G \cong H$ if and only if for each p dividing n, $G(p) \cong H(p)$.

4. Let G, H be finite abelian groups. Prove that $G \cong H$ if and only if for each $n \in \mathbb{N}$, G and H have the same number of elements of order n.

5. Let p be prime. Prove that the groups $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$ are not isomorphic.

6. Let $U = \{x \in \mathbb{C} : |x| = 1\}$. This forms an abelian group under multiplication. Find a subgroup $H \leq U$ such that $H \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

7. Calculate the betti number, elementary divisors, and invariant factors of $\mathbb{Z}_{10} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{21}$.

8. Calculate the betti number, elementary divisors, and invariant factors of $\mathbb{Z} \oplus \mathbb{Z}_6 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{10}$.

9. Calculate the betti number, elementary divisors, and invariant factors of $\mathbb{Z}_{54000}^{\times}$.

10. Find the Smith Normal Form of $\begin{bmatrix} 4 & 2 & 4 \\ 2 & 2 & 2 \\ 6 & 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 4 & 2 & 4 \\ 2 & 2 & 2 \\ 6 & 5 & 6 \end{bmatrix}$, and $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 2 & 2 \\ 6 & 0 & 6 \end{bmatrix}$.

11. Find all finite abelian groups, of order at most 50, of rank 2.

12. Find all finite abelian groups, of order at most 50, of rank 3.

13. Find all finite abelian groups, of order at most 100, whose elementary divisors form a decreasing sequence.

14. Find all nonisomorphic abelian groups, that are generated by at most two elements.

15. Find all nonisomorphic abelian groups, whose rank and exponent are both 4.

16. For $G = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$, find generators (elements within G) and a set of relations on these generators, that fully characterizes G.

17. For $G = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$, find a subgroup $H \leq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ such that $G \cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})/H$.

18. Describe the relationship between the ideal class group of a ring, and factorization into irreducibles in that ring.

19. Find all irreducibles in $\mathcal{B}(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$. What is the Davenport constant?

20. What is the Davenport constant of $\mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$? Find an irreducible, of that size, in the block monoid.