MATH 521B: Abstract Algebra

Exam 3 Solutions

1. Calculate the elementary divisors and invariant factors of $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{45}$.

We first break up the cyclic groups into prime powers, as $\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$. We can then read off the elementary divisors as $2, 4, 3, 9, 5, 5$. Arranging as $\begin{smallmatrix} 2 & 4 \\ 3 & 9 \\ 5 & 5 \end{smallmatrix}$ gives the invariant factors as $30, 180$.

2. Find all nonisomorphic finite abelian groups, of order between 40 and 80 inclusive, of rank 2. How many are there?

All such groups will be of the form $\mathbb{Z}_a \oplus \mathbb{Z}_{ab}$, for $a \geq 2$ and $b \geq 1$. The problem conditions specify that $40 \leq a^2 b \leq 80$; hence $a < 9$ since $9^2 > 80$. All possible values:

a:	2	3	4	5	6	7	8
b:	$10, 11, \ldots, 20$	$5, 6, 7, 8$	$3, 4, 5$	$2, 3$	2	1	1

Explicitly, these 23 groups are: $\mathbb{Z}_2 \oplus \mathbb{Z}_{20}, \mathbb{Z}_2 \oplus \mathbb{Z}_{22}, \ldots, \mathbb{Z}_2 \oplus \mathbb{Z}_{40}, \mathbb{Z}_3 \oplus \mathbb{Z}_{15}, \mathbb{Z}_3 \oplus \mathbb{Z}_{18}, \mathbb{Z}_3 \oplus \mathbb{Z}_{21}, \mathbb{Z}_3 \oplus \mathbb{Z}_{24}, \mathbb{Z}_4 \oplus \mathbb{Z}_{12}, \mathbb{Z}_4 \oplus \mathbb{Z}_{16}, \mathbb{Z}_4 \oplus \mathbb{Z}_{20}, \mathbb{Z}_5 \oplus \mathbb{Z}_{10}, \mathbb{Z}_5 \oplus \mathbb{Z}_{15}, \mathbb{Z}_6 \oplus \mathbb{Z}_{12}, \mathbb{Z}_7 \oplus \mathbb{Z}_7, \mathbb{Z}_8 \oplus \mathbb{Z}_8$.

3. Find all nonisomorphic abelian groups, of rank 3, of exponent 27. How many are there?

We have $G \cong \mathbb{Z}_{3^a} \oplus \mathbb{Z}_{3^b} \oplus \mathbb{Z}_{3^3}$, where $1 \leq a \leq b \leq 3$. These have possible values $(a, b) \in \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$, leading to the six groups $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{27}, \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{27}, \mathbb{Z}_3 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_{27}, \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{27}, \mathbb{Z}_9 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_{27}, \mathbb{Z}_{27} \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_{27}$.

4. For $G = \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$, find the Davenport constant $D(G)$. Also, find an irreducible with $D(G)$ elements, in the block monoid $\mathcal{B}(G)$.

We first write G in invariant factor form, i.e. $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{30}$. Since this is rank 2, it is known that $D(G) = D^\star(G) = (2-1) + (30-1) + 1 = 31$. An irreducible of that size is $(1,0)^1(0,1)^{29}(1,1)^1$.

5. For $G = \mathbb{Z}_3 \oplus \mathbb{Z}_6$, determine $G(2)$ and $G(3)$ explicitly.

We have $|G| = 18 = 2 \cdot 9$, so $G(2) = \{g \in G : |g| = 1 \text{ or } 2\} = \{(0,0), (0,3)\}$. Also, $G(3) = \{g \in G : |g| = 1, 3, \text{ or } 9\}$, however since $exp(G) = 6$, no elements of order 9 exist. We have $G(3) = \{(0,0), (1,0), (2,0), (0,2), (1,2), (2,2), (0,4), (1,4), (2,4)\}$.

6. Find all nonisomorphic abelian groups, that are generated by at most two elements.

We apply the Fundamental Theorem of Finitely Generated Abelian Groups, and classify these by betti number. If 2, the only group is $\mathbb{Z} \oplus \mathbb{Z}$. If 1, the 2-generated groups are $\mathbb{Z} \oplus \mathbb{Z}_n$, for each integer $n \geq 2$; the sole 1-generated group is $\mathbb{Z}$. Lastly, if 0, the 2-generated groups are $\mathbb{Z}_a \oplus \mathbb{Z}_{ab}$, for all integers a, b satisfying $a \geq 2$ and $b \geq 1$; the 1-generated groups are $\mathbb{Z}_n$, for each integer $n \geq 2$; and the 0-generated (trivial) group is $\{0\}$.

7. Suppose G, H are both finitely generated abelian groups with $G \cong H$. Prove that G, H have the same betti numbers.

By the Fundamental Theorem of Finitely Generated Abelian Groups, $G \cong \mathbb{Z}^a \oplus G'$, $H \cong \mathbb{Z}^b \oplus H'$, where a, b are the betti numbers of G, H (respectively), and G', H' are torsion. Without loss of generality, suppose that $a > b$. Let $\{g_1, g_2, \ldots, g_a\}$ be a set of generators for $\mathbb{Z}^a$. Let $\phi : (\mathbb{Z}^a \oplus G') \to (\mathbb{Z}^b \oplus H')$ be an isomorphism. Then $\{\phi(g_1), \ldots, \phi(g_a)\}$ are all torsion-free; hence must be elements of $\mathbb{Z}^b$. These are linearly dependent over $\mathbb{Q}$, since they are elements of the b-dimensional vector space $\mathbb{Q}^b$, i.e. there are rational coefficients to make $\alpha_1\phi(g_1) + \cdots + \alpha_a\phi(g_a) = 0$. We can multiply by the lcm of the denominators to make the coefficients integers. This is a contradiction, as $\mathbb{Z}^b$ is free.

8. Let $U = \{x \in \mathbb{C} : |x| = 1\}$. This forms an abelian group under multiplication. Find a subgroup $H \leq U$ such that $H \cong \mathbb{Z} \oplus \mathbb{Z}$.

We need two numbers that are algebraically independent from each other, and also from 2π. For example, $\sqrt{2}$ and 1. This gives $g = e^{\sqrt{2}i}, h = e^i$. Set $H = \langle g, h \rangle$. We have $g^a h^b = e^{(a\sqrt{2}+b)i}$. We now prove that H is torsion-free. If $g^a h^b = 1 = e^{2k\pi i}$, then $(a\sqrt{2} + b)i = 2k\pi i$ for some integers a, b, k. If $k \neq 0$, then we divide both sides by $2ki$, which gives the transcendental number π equal to an algebraic number, which is a contradiction. Hence $k = 0$. If $a \neq 0$, we divide by i and rearrange to get $\sqrt{2} = \frac{-b}{a}$, which is again a contradiction since $\sqrt{2}$ is irrational. Hence $a = b = k = 0$, and H is torsion-free. By the Fundamental Theorem of Finitely Generated Abelian Groups, the only torsion-free abelian group with two generators is (isomorphic to) $\mathbb{Z} \oplus \mathbb{Z}$.