MATH 521B: Abstract Algebra
Homework 10: Due Apr. 20

For problems 1-5, find all abelian groups (up to isomorphism) of the specified order. For each group, give its elementary divisors and invariant factors.

1. Order 25.

2. Order 100.

3. Order 500.

4. Order 1800.

5. Order 2730.

Recall that $\mathbb{Z}_n^\times$ is the multiplicative group of units modulo n, while $\mathbb{Z}_n$ is the additive group modulo n.

6. Write the multiplication table for $\mathbb{Z}_8^\times$, and use this to prove that $\mathbb{Z}_8^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

7. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, a prime decomposition. Prove that $\mathbb{Z}_n^\times \cong \mathbb{Z}_{p_1^{a_1}}^\times \times \mathbb{Z}_{p_2^{a_2}}^\times \times \cdots \times \mathbb{Z}_{p_k^{a_k}}^\times$.

For the following, you may assume the following: (a) $\mathbb{Z}_2^\times \cong \mathbb{Z}_1$, $\mathbb{Z}_4^\times \cong \mathbb{Z}_2$; (b) For $k \geq 3$, $\mathbb{Z}_{2^k}^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{k-2}}$; and (c) For any odd prime p, $\mathbb{Z}_{p^k}^\times \cong \mathbb{Z}_t$, for $t = p^k - p^{k-1}$.

8. Find the elementary divisors and invariant factors of $\mathbb{Z}_{12}^\times$.

9. Find the elementary divisors and invariant factors of $\mathbb{Z}_{24}^\times$.

10. Find the elementary divisors and invariant factors of $\mathbb{Z}_{56}^\times$.

11. Find the elementary divisors and invariant factors of $\mathbb{Z}_{60}^\times$.

12. Find the elementary divisors and invariant factors of $\mathbb{Z}_{63}^\times$.

13. Find the elementary divisors and invariant factors of $\mathbb{Z}_{1100}^\times$.

14. Find the elementary divisors and invariant factors of $\mathbb{Z}_{3600}^\times$.

15. Prove that $\mathbb{Q}$, the set of rationals under addition, is not finitely generated.

For problems 16-17, let $G = \{(a, b) : a \equiv b \pmod{10}\}$, a subset of $\mathbb{Z} \times \mathbb{Z}$.

16. Prove that $G \leq \mathbb{Z} \times \mathbb{Z}$, and that G is torsion-free.

17. Prove that $G \cong \mathbb{Z} \times \mathbb{Z}$.