MATH 579: Combinatorics
Homework 10 Solutions

1. In S_6, set $\pi = \left(\begin{smallmatrix}1&2&3&4&5&6\\4&6&5&3&1&2\end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix}1&2&3&4&5&6\\3&6&2&1&5&4\end{smallmatrix}\right)$. Calculate $\pi\circ\pi, \pi\circ\pi\circ\pi, \pi\circ\tau, \tau\circ\pi, \tau\circ\tau$.

   We have $\pi = (1,4,3,5)(2,6)$, so $\pi\circ\pi = (1,3)(4,5)$, and $\pi\circ\pi\circ\pi = (1,5,3,4)(2,6)$. We have $\tau = (1,3,2,6,4)$, so $\pi\circ\tau = (1,5)(3,6)$ while $\tau\circ\pi = (2,4)(3,5)$ and $\tau\circ\tau = (1,2,4,3,6)$.

2. In S_6, set $\pi = \left(\begin{smallmatrix}1&2&3&4&5&6\\4&6&5&3&1&2\end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix}1&2&3&4&5&6\\3&6&2&1&5&4\end{smallmatrix}\right)$. Calculate π^{-1} and τ^{-1}. Express each answer in both two-line notation and in cycle notation.

   In cycle notation, finding inverses is easy, just reverse things. $\pi^{-1} = (6,2)(5,3,4,1) = (1,5,3,4)(2,6)$ and $\tau^{-1} = (4,6,2,3,1) = (1,4,6,2,3)$. We can then convert to two-line notation as $\pi^{-1} = \left(\begin{smallmatrix}1&2&3&4&5&6\\5&6&4&1&3&2\end{smallmatrix}\right)$ and $\tau^{-1} = \left(\begin{smallmatrix}1&2&3&4&5&6\\4&3&1&6&5&2\end{smallmatrix}\right)$.

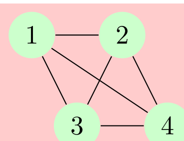
3. In S_6, set $\pi = \left(\begin{smallmatrix}1&2&3&4&5&6\\4&6&5&3&1&2\end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix}1&2&3&4&5&6\\3&6&2&1&5&4\end{smallmatrix}\right)$. Find all elements of $\langle\pi\rangle$ and $\langle\tau\rangle$.

   π has cycles of length $2,4$, so $\langle\pi\rangle$ will have $\mathrm{lcm}(2,4) = 4$ elements, namely $\pi, \pi^2 = \pi\circ\pi, \pi^{-1}(=\pi^3)$, and the identity. All four were calculated above. τ has a single cycle of length 5, so $\langle\tau\rangle$ will have five elements, namely $\tau, \tau^2 = \tau\circ\tau, \tau^3, \tau^4 = \tau^{-1}$, and the identity. Four of these five are above; the missing is $\tau^3 = \tau\circ\tau^2 = (1,6,3,4,2)$.

4. In S_4, set $\gamma = (1,2,3,4)$ and $\rho = (1,3)$. Find all elements of $\langle\gamma,\rho\rangle$.

   We first calculate $\langle\gamma\rangle$, which has the four elements: $\gamma, \gamma^2 = (1,3)(2,4), \gamma^3 = (1,4,3,2)$, and the identity id. We can multiply by ρ on the right, to get $\gamma\circ\rho = (1,4)(2,3)$, $\gamma^2\circ\rho = (2,4)$, $\gamma^3\circ\rho = (1,2)(3,4)$, and of course $id\circ\rho = \rho$. Multiplying by ρ on the right again doesn't gain anything, since $\rho\circ\rho = id$. It turns out that these eight elements are all you can get; they are the group of rotations of the square, where γ is a 90° rotation, and ρ is a reflection through a diagonal line. This group is called the dihedral group on four elements, D_4.

5. Find the automorphism group for:

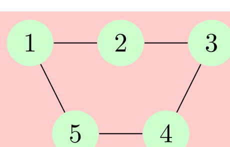


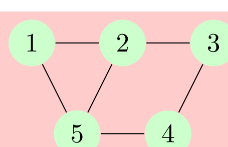
   We have complete symmetry here, any permutation will be an automorphism. Hence the answer is S_4.

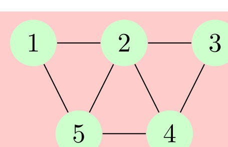
6. Find the group of rotations for the (solid) tetrahedron, and contrast with the answer to the previous problem.

   Let's use the labelling of problem 5, but this time for a solid shape. Note that $(1,2)$ is an automorphism of the graph but not a rotation of the solid. To achieve this permutation, one would need to turn the tetrahedron inside-out (which can be done if it's a wireframe, or using the fourth dimension, but not as a rotation). It turns out that just half of the $24 = |S_4|$ automorphisms are impossible in this manner. The twelve possible rotations are: $(2,3,4), (2,4,3), (1,3,4), (1,4,3), (1,2,4), (1,4,2), (2,3,4), (2,4,3), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), id$. The first eight correspond to rotations around an axis that passes through one vertex and the opposite face. The next three correspond to rotations around an axis that passes through two "opposite" edges. This group is called the alternating group on four elements, A_4.

7. Find the aut. groups for:



,


,


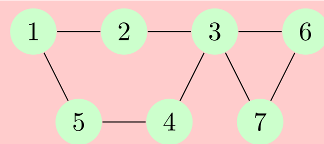
.

   The first one is the dihedral group on five elements, D_5, and is generated by a rotation, e.g.

$(1,2,3,4,5)$ and a reflection, e.g. $(2,5)(3,4)$. Written out explicitly, the elements are id, $(1, 2, 3, 4, 5)$, $(1, 3, 5, 2, 4)$, $(1, 4, 2, 5, 3)$, $(1, 5, 4, 3, 2)$, $(2, 5)(3, 4)$, $(1, 2)(3, 5)$, $(1, 3)(4, 5)$, $(1, 4)(2, 3)$, $(1, 5)(2, 4)$.

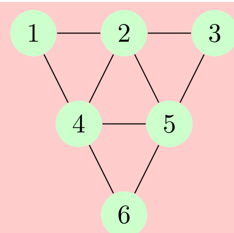
The second one is easier; 1 must be fixed, so either $2, 5$ are fixed (and we get id), or $2, 5$ swap, and we get $(2, 5)(3, 4)$. Similarly for the third one; 2 must be fixed, so either $1, 3$ are fixed (and we get id), or $1, 3$ swap, and we get $(1, 3)(4, 5)$.

8. Find the automorphism group for:



The vertex 3 must be fixed, so we independently either fix or swap $6, 7$, and fix or swap $2, 4$. This gives four group elements: $id, (6, 7), (2, 4)(1, 5), (2, 4)(1, 5)(6, 7)$. All four elements are their own inverse; this is (a version of) the Klein 4-group.

9. Find the automorphism group for:



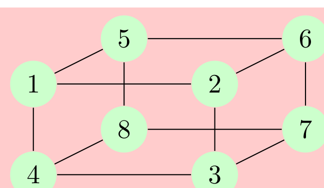
The vertices $2, 4, 5$ don't affect the action – this is the "same" automorphism group as for the triangle. Each vertex goes to a vertex, and we either flip the triangle over or we don't. Hence there are $6 = |S_3|$ permutations: $(1, 3, 6)(2, 5, 4), (1, 6, 3)(2, 4, 5)$, $(1, 6)(2, 5), (1, 3)(4, 5), (3, 6)(2, 4), id$. The first two correspond to rotations, the next three to reflections.

10. Find the group of rotations for a (solid) cube.

Let's use the labelling from problem 11. The axis of rotation can come in three types. It can pass through two opposite faces, giving a permutation of order 4: $(1, 2, 3, 4)(5, 6, 7, 8)$, $(1, 3)(2, 4)(5, 7)(6, 8)$, $(1, 4, 3, 2)(5, 8, 7, 6)$, and $(1, 5, 8, 4)(2, 6, 7, 3)$, $(1, 8)(4, 5)(2, 7)(3, 6)$, $(1, 4, 8, 5)(2, 3, 7, 6)$, and $(1, 2, 6, 5)(3, 7, 8, 4)$, $(1, 6)(2, 5)(3, 8)(4, 7)$, $(1, 5, 6, 2)(3, 4, 8, 7)$. It can pass through opposite corners, giving a permutation of order 3: $(2, 4, 5)(3, 8, 6)$, $(2, 5, 4)(3, 6, 8)$ and $(1, 3, 6)(4, 7, 5)$, $(1, 6, 3)(4, 5, 7)$ and $(1, 6, 8)(2, 7, 4)$, $(1, 8, 6)(2, 4, 7)$ and $(1, 3, 8)(2, 7, 5)$, $(1, 8, 3)(2, 5, 7)$. It can pass through the centers of two opposite edges, giving a permutation of order 2: $(1, 2)(3, 5)(4, 6)(7, 8)$, $(1, 7)(2, 8)(3, 4)(5, 6)$, $(1, 5)(2, 8)(3, 7)(4, 6)$, $(1, 7)(2, 6)(3, 5)(4, 8)$, $(1, 4)(2, 8)(3, 5)(6, 7)$, $(1, 7)(2, 3)(4, 6)(5, 8)$. Lastly, there is id, giving 24 in all.

It turns out that this is the "same" group as S_4, which can be proved by considering the pairs $\{1, 7\}, \{2, 8\}, \{3, 5\}, \{4, 6\}$. Every rotation maps one of these four pairs to another, and every permutation of the four pairs is possible.

11. Find the automorphism group for:



Contrast with the previous problem.

Similarly to the tetrahedron case, if the cube isn't solid (wireframe, or we can use the 4th dimension) we can do an "eversion", which turns the cube inside out. This is the permutation $e = (1, 7)(2, 8)(3, 5)(4, 6)$. We then get 48 permutations – the 24 from problem 10, and 24 more, composing those permutations with e.