MATH 579: Combinatorics
Homework 3 Solutions

1. Let $n \in \mathbb{N}$ be arbitrary. Determine the number of solutions in integers to $x_1+x_2+x_3+\cdots+x_n = 26$, with each $x_i \geq 0$. Your answer should be a function of n.
   These may be considered as multisets $\{1^{a_1}2^{a_2}\cdots n^{a_n}\}$, where $a_1 + a_2 + \cdots + a_n = 26$. Hence there are $\left(\!\binom{n}{26}\!\right) = \binom{n+25}{26}$ solutions.

2. Let $n \in \mathbb{N}$ be arbitrary. Determine the number of solutions in integers to $x_1+x_2+x_3+\cdots+x_n = 26$, with each $x_i \geq -2$. Your answer should be a function of n.
   We make the substitution $x_i = -2 + x_i'$. This changes the problem to $x_1' + \cdots + x_n' = 26 + 2n$, and we need $x_i' \geq 0$. Applying problem 1, we get $\left(\!\binom{n}{26+2n}\!\right) = \binom{25+3n}{26+2n}$ solutions.

3. Let $n \in \mathbb{N}$ be arbitrary. Determine the number of solutions in integers to $x_1+x_2+x_3+\cdots+x_n = 26$, with each $x_i \geq 1+i$. Your answer should be a function of n.
   We make the substitution $x_i = 1+i+x_i'$. This changes the problem to $x_1'+\cdots+x_n' = 26-\sum_{i=1}^{n}(1+i) = 26 - n - \frac{n(n+1)}{2} = \frac{-n^2-3n+52}{2}$. If $n \geq 6$ the RHS turns out negative, so there are no solutions. For $n < 6$, we set $k = \frac{-n^2-3n+52}{2}$ and apply problem 1 to get $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ solutions.

4. Let $n \in \mathbb{N}$ be arbitrary. Determine the number of solutions in integers to $x_1+x_2+x_3+\cdots+x_n = 26$, with each $x_i \geq i^2$. Your answer should be a function of n.
   We make the substitution $x_i = i^2 + x_i'$. This changes the problem to $x_1' + \cdots + x_n' = 26 - \sum_{i=1}^{n} i^2 = 26 - \frac{n(n+1)(2n+1)}{6}$. For $n \geq 4$, the RHS turns out negative, so there are no solutions. For $n < 4$, we set $k = 26 - \frac{n(n+1)(2n+1)}{6}$ and apply problem 1 to get $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ solutions.

5. Find explicitly all integer partitions of 8. Match each partition with its conjugate, and give its rank.
   There should be $p(8) = 22$ integer partitions. There are eight rank-one partitions: 8 matched with $1+1+1+1+1+1+1+1$, $7+1$ matched with $2+1+1+1+1+1+1$, $6+1+1$ matched with $3+1+1+1+1+1$, $5+1+1+1$ matched with $4+1+1+1+1$. The rest are rank-two partitions: $6+2$ matched with $2+2+1+1+1+1$, $5+3$ matched with $2+2+2+1+1$, $4+4$ matched with $2+2+2+2$, $5+2+1$ matched with $3+2+1+1+1$, $4+2+2$ matched with $3+3+1+1$, $4+3+1$ matched with $3+2+2+1$. There are two self-conjugate partitions, $3+3+2$ and $4+2+1+1$.

6. Find explicitly all integer partitions of 10 into odd parts.
   There should be 10 altogether. Five are rank-one: $9+1$, $7+1+1+1$, $5+1+1+1+1+1$, $3+1+1+1+1+1+1+1$, $1+1+1+1+1+1+1+1+1+1$. Four are rank-two: $7+3$, $5+5$, $5+3+1+1$, $3+3+1+1+1+1$. One is rank-three: $3+3+3+1$.

7. Find explicitly all integer partitions of 10 into distinct parts.
   There is a famous theorem, due to Euler, that the number of answers to this problem and the last are the same. Alas, there is no cute bijection here, so we just list the ten solutions: 10, $9+1$, $8+2$, $7+3$, $6+4$, $7+2+1$, $6+3+1$, $5+3+2$, $5+4+1$, $4+3+2+1$.

8. Find explicitly all integer partitions of 25 into distinct odd parts.
   Note that there must be an odd number of terms, since 25 is odd and each term is odd. There are twelve such integer partitions altogether: 25, $21+3+1$, $19+5+1$, $17+7+1$, $17+5+3$, $15+9+1$, $15+7+3$, $13+11+1$, $13+9+3$, $13+7+5$, $11+9+5$, $9+7+5+3+1$.

9. Find explicitly all self-conjugate integer partitions of 25, and match them with your answers from exercise 8.

We take each partition and turn each term into a (self-conjugate) hook. In order, we get $13+1+1+1+1+1+1+1+1+1+1+1+1$, $11+3+3+1+1+1+1+1+1+1+1$, $10+4+3+2+1+1+1+1+1+1$, $9+5+3+2+2+1+1+1+1$, $9+4+4+3+1+1+1+1+1$, $8+6+3+2+2+2+1+1$, $8+5+4+3+2+1+1+1$, $7+7+3+2+2+2+2$, $7+6+4+3+2+2+1$, $7+5+5+3+3+1+1$, $6+6+5+3+3+2$, $5+5+5+5+5$.

10. Prove that $p(1)+p(2)+\cdots+p(n) < p(2n)$ for all $n \in \mathbb{N}$.

Consider all partitions of $2n$, with largest term k. Set aside those with $k < n$; this explains the strict inequality. Removing that term k leaves a partition of $2n-k \leq n$. On the other hand, given any partition of i with $1 \leq i \leq n$, we add an additional term of $2n-i$ to get a partition of $2n$; the term we added must be the largest of the resulting partition. Hence we have a bijection between partitions of $1, 2, \ldots, n$ and those partitions of $2n$ whose largest part is at least n.

11. Let $v(n)$ denote the number of integer partitions of n in which each part is at least 2. Prove that $v(n) = p(n) - p(n-1)$, for all $n \geq 2$.

Consider partitions of n. Some have smallest term 1, while the rest have smallest term at least 2. We count the former by removing that smallest term, which leaves a partition of $n-1$, counted by $p(n-1)$. We count the latter with $v(n)$. Hence $p(n) = v(n) + p(n-1)$, as desired.

12. Let $n, k \in \mathbb{N}$. Let $p_k(n)$ denote the number of integer partitions of n into exactly k parts. Prove that $p_k(n)$ also counts the number of partitions of n, in which the largest part has size k.

Conjugation swaps the statistics "number of parts" and "maximum size of a part" in a partition. Hence it is a bijection between partitions with exactly k parts, and those with maximum size of a part equal to k.

13. Prove that $p_k(n)$ satisfies the recurrence relation $p_k(n) = p_k(n-k) + p_{k-1}(n-1)$.

Consider partitions of n into exactly k parts. Some have smallest part 1, while the rest have smallest part ≥ 1. We count the former by deleting that smallest part, leaving a partition of $n-1$ into $k-1$ parts. We count the latter, by reducing each of the k parts by 1. The result will leave k nonempty parts, so we have a partition of $n-k$ into k parts.

14. Using the recurrence relation from exercise 13, and the boundary conditions $p_1(n) = 1$ ($\forall n \in \mathbb{N}$) and $p_k(n) = 0$ ($\forall k \in \mathbb{N}, \forall n \in \mathbb{Z}$ with $n < k$), calculate $p_5(10)$.

$p_5(10) = p_5(5) + p_4(9)$. We have insight into $p_5(5)$, but that wouldn't use the method specified, so instead we calculate: $p_5(5) = p_5(0) + p_4(4) = p_4(4) = p_4(0) + p_3(3) = p_3(3) = p_3(0) + p_2(2) = p_2(2) = p_2(0) + p_1(2) = 0 + 1 = 1$.

We calculate $p_4(9) = p_4(5) + p_3(8)$. We calculate $p_4(5) = p_4(1) + p_3(4) = p_3(4) = p_3(1) + p_2(3) = p_2(3) = p_2(1) + p_1(2) = 0 + 1 = 1$. Hence $p_5(10) = p_5(5) + p_4(5) + p_3(8) = 2 + p_3(8)$.

We calculate $p_3(8) = p_3(5) + p_2(7)$. We have $p_3(5) = p_3(2) + p_2(4) = p_2(4) = p_2(2) + p_1(3) = 1 + p_2(2) = 1 + p_2(0) + p_1(1) = 1 + 0 + 1 = 2$. Hence $p_5(10) = 2 + p_3(8) = 4 + p_2(7)$.

We have $p_2(7) = p_2(5) + p_1(6) = 1 + p_2(5) = 1 + p_2(3) + p_1(4) = 2 + p_2(3) = 2 + p_2(1) + p_1(4) = 2 + 0 + 1 = 3$. Putting it all together, we have $p_5(10) = 4 + 3 = 7$.