MATH 579: Combinatorics

Homework 9 Solutions

1. Find the number of integers in $[1, 1000]$ relatively prime to 70.

We have $70 = 2 \cdot 5 \cdot 7$, so $S = \{s_2, s_5, s_7\}$. We have $f_=(\emptyset) = f_\geq(\emptyset) - f_\geq(s_2) - f_\geq(s_5) - f_\geq(s_7) + f_\geq(s_2 \cup s_5) + f_\geq(s_2 \cup s_7) + f_\geq(s_5 \cup s_7) - f_\geq(S) = 1000 - \lfloor\frac{1000}{2}\rfloor - \lfloor\frac{1000}{5}\rfloor - \lfloor\frac{1000}{7}\rfloor + \lfloor\frac{1000}{10}\rfloor + \lfloor\frac{1000}{14}\rfloor + \lfloor\frac{1000}{35}\rfloor - \lfloor\frac{1000}{70}\rfloor = 1000 - 500 - 200 - 142 + 100 + 71 + 28 - 14 = 343$.

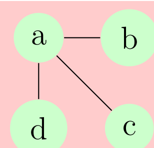
2. Find the number of integers in $[1, 720]$ relatively prime to 720.

We have $720 = 2^4 \cdot 3^2 \cdot 5$, so $S = \{s_2, s_3, s_5\}$. We have $f_=(\emptyset) = f_\geq(\emptyset) - f_\geq(s_2) - f_\geq(s_3) - f_\geq(s_5) + f_\geq(s_2 \cup s_2) + f_\geq(s_2 \cup s_5) + f_\geq(s_3 \cup s_5) - f_\geq(S) = 720 - \lfloor\frac{720}{2}\rfloor - \lfloor\frac{720}{3}\rfloor - \lfloor\frac{720}{5}\rfloor + \lfloor\frac{720}{6}\rfloor + \lfloor\frac{720}{10}\rfloor + \lfloor\frac{720}{15}\rfloor - \lfloor\frac{720}{30}\rfloor = 720 - 360 - 240 - 144 + 120 + 72 + 48 - 24 = 192$.

3. Find the number of integers in $[1, 1000]$ relatively prime to 210.

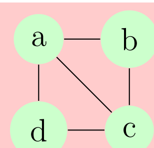
We have $210 = 2 \cdot 3 \cdot 5 \cdot 7$, so $S = \{s_2, s_3, s_5, s_7\}$. We have $f_=(\emptyset) = f_\geq(\emptyset) - f_\geq(s_2) - f_\geq(s_2) - f_\geq(s_5) - f_\geq(s_7) + f_\geq(s_2 \cup s_3) + f_\geq(s_2 \cup s_5) + f_\geq(s_2 \cup s_7) + f_\geq(s_3 \cup s_5) + f_\geq(s_3 \cup s_7) + f_\geq(s_5 \cup s_7) - f_\geq(s_2 \cup s_3 \cup s_5) - f_\geq(s_2 \cup s_3 \cup s_7) - f_\geq(s_2 \cup s_5 \cup s_7) - f_\geq(s_3 \cup s_5 \cup s_7) + f_\geq(S) = 1000 - \lfloor\frac{1000}{2}\rfloor - \lfloor\frac{1000}{3}\rfloor - \lfloor\frac{1000}{5}\rfloor - \lfloor\frac{1000}{7}\rfloor + \lfloor\frac{1000}{6}\rfloor + \lfloor\frac{1000}{10}\rfloor + \lfloor\frac{1000}{14}\rfloor + \lfloor\frac{1000}{15}\rfloor + \lfloor\frac{1000}{21}\rfloor + \lfloor\frac{1000}{35}\rfloor - \lfloor\frac{1000}{30}\rfloor - \lfloor\frac{1000}{42}\rfloor - \lfloor\frac{1000}{70}\rfloor - \lfloor\frac{1000}{105}\rfloor + \lfloor\frac{1000}{210}\rfloor = 1000 - 500 - 333 - 200 - 142 + 166 + 100 + 71 + 66 + 47 + 28 - 33 - 23 - 14 - 9 + 4 = 228$.

4. Find the chromatic polynomial for:



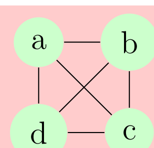
We have $S = \{ab, ac, ad\}$, so $f_=(\emptyset) = f_\geq(\emptyset) - f_\geq(ab) - f_\geq(ac) - f_\geq(ad) + f_\geq(abc) + f_\geq(abd) + f_\geq(acd) - f_\geq(abcd) = x^4 - 3x^3 + 3x^2 - x$.

5. Find the chromatic polynomial for:



We have $S = \{ab, ac, ad, bc, cd\}$, so $f_=(\emptyset) = f_\geq(\emptyset) - \binom{5}{1} f_\geq(ab) + \binom{5}{2} f_\geq(abc) - f_\geq(abc) - f_\geq(acd) - 8 f_\geq(abcd) + 5 f_\geq(abcd) - f_\geq(abcd) = x^4 - 5x^3 + 10x^2 - 2x^2 - 8x + 5x - x = x^4 - 5x^3 + 8x^2 - 4x$.

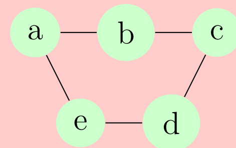
6. Find the chromatic polynomial for:



There are six edges, so $|S| = 6$. We will use symmetry, to avoid having a very hard time. $f_\geq(\emptyset) = x^4$, $f_\geq(1 \text{ edge}) = x^3$, $f_\geq(2 \text{ edges}) = x^2$, $f_\geq(4 \text{ or more edges}) = x$. The only tricky bit is with 3 edges. If the three edges form one of the four triangles abd, abc, bcd, or acd, then $f_\geq(3 \text{ edges}) = x^2$. However for all the other $\binom{6}{3} - 4 = 16$

choices of three edges, $f_{\geq}(3\text{ edges}) = x$. Hence $f_{=}(\emptyset) = x^4 - \binom{6}{1}x^3 + \binom{6}{2}x^2 - 4x^2 - 16x + \binom{6}{4}x - \binom{6}{5}x + \binom{6}{6}x = x^4 - 6x^3 + 11x^2 - 6x$. This equals $x(x-1)(x-2)(x-3)$, which is not a coincidence.

7. Find the chromatic polynomial for:



We have $S = \{ab, bc, cd, de, ea\}$. Note that this graph is triangle-free, so $f_{\geq}(3\text{ edges}) = x^2$. Further, any four edges form a spanning tree, so $f_{\geq}(4\text{ edges}) = x$. Hence, $f_{=}(\emptyset) = f_{\geq}(\emptyset) - \binom{5}{1}f_{\geq}(1\text{ edge}) + \binom{5}{2}f_{\geq}(2\text{ edges}) - \binom{5}{3}f_{\geq}(3\text{ edges}) + \binom{5}{4}f_{\geq}(4\text{ edges}) - \binom{5}{5}f_{\geq}(5\text{ edges}) = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - x = x^5 - 5x^4 + 10x^3 - 10x^2 + 4x$.

8. Determine the number of solutions, in nonnegative integers, to $a + b + c + d = 30$, where $a \leq 10$, $b \leq 11$, and $c \leq 12$.

We have $S = \{s_a, s_b, s_c\}$, where s_a means $a \geq 11$, s_b means $b \geq 12$, and s_c means $c \geq 13$. For example, note that $f_{\geq}(s_a) = \left(\!\binom{4}{19}\!\right)$, because we are solving $a+b+c+d = 30$ with $a \geq 11$; so by taking $a' = a - 11$ we count solutions to $a'+b+c+d = 19$ in $\mathbb{N}_0$. So, $f_{=}(\emptyset) = f_{\geq}(\emptyset) - f_{\geq}(s_a) - f_{\geq}(s_b) - f_{\geq}(s_c) + f_{\geq}(s_a s_b) + f_{\geq}(s_a s_c) + f_{\geq}(s_b s_c) - f_{\geq}(s_a s_b s_c) = \left(\!\binom{4}{30}\!\right) - \left(\!\binom{4}{19}\!\right) - \left(\!\binom{4}{18}\!\right) - \left(\!\binom{4}{17}\!\right) + \left(\!\binom{4}{7}\!\right) + \left(\!\binom{4}{6}\!\right) + \left(\!\binom{4}{5}\!\right) - 0 = 1706$.

9. Determine the number of solutions, in nonnegative integers, to $a + b + c + d = 30$, where $a \leq 10$, $b \leq 10$, and $c \leq 10$.

. We have $S = \{s_a, s_b, s_c\}$, where s_a means $a \geq 11$, s_b means $b \geq 11$, and s_c means $c \geq 11$. So, $f_{=}(\emptyset) = f_{\geq}(\emptyset) - 3f_{\geq}(s_a) + 3f_{\geq}(s_a s_b) - f_{\geq}(s_a s_b s_c) = \left(\!\binom{4}{30}\!\right) - 3\left(\!\binom{4}{19}\!\right) + 3\left(\!\binom{4}{8}\!\right) = 1331$.

10. Determine the number of solutions, in nonnegative integers, to $a + b + c + d = 30$, where $3 \leq a \leq 10$, $2 \leq b \leq 11$, and $1 \leq c \leq 12$.

We start with the substitutions $a' = a - 3, b' = b - 2, c' = c - 1$, which changes the problem to $a' + b' + c' + d = 24$, where $a' \leq 7, b' \leq 9, c' \leq 11$. We have $S = \{s_{a'}, s_{b'}, s_{c'}\}$, where $s_{a'}$ means $a' \geq 8$, $s_{b'}$ means $b' \geq 10$, and $s_{c'}$ means $c' \geq 12$. So, $f_{=}(\emptyset) = f_{\geq}(\emptyset) - f_{\geq}(s_{a'}) - f_{\geq}(s_{b'}) - f_{\geq}(s_{c'}) + f_{\geq}(s_{a'}s_{b'}) + f_{\geq}(s_{a'}s_{c'}) + f_{\geq}(s_{b'}s_{c'}) - f_{\geq}(s_{a'}s_{b'}s_{c'}) = \left(\!\binom{4}{24}\!\right) - \left(\!\binom{4}{16}\!\right) - \left(\!\binom{4}{14}\!\right) - \left(\!\binom{4}{12}\!\right) + \left(\!\binom{4}{6}\!\right) + \left(\!\binom{4}{4}\!\right) + \left(\!\binom{4}{2}\!\right) - 0 = 950$.

11. Determine the number of l.o.d.e.'s of length 10, drawn from [10], where exactly two integers are in their natural position.

Let's count how many l.o.d.e.'s have the last two integers in their natural position. Then, the remaining eight integers are a derangement (i.e. none are in their natural position). There are $D_8 = 8! - \binom{8}{1}7! + \binom{8}{2}6! - \binom{8}{3}5! + \binom{8}{4}4! - \binom{8}{5}3! + \binom{8}{6}2! - \binom{8}{7}1! + \binom{8}{8}0! = 14833$. But now, we could instead have chosen any of the $\binom{10}{2} = 45$ pairs of elements to be in their natural position. We can multiply these, as none of these 45 classes of l.o.d.e.'s have any elements in common. Hence our answer is 667485.