MATH 623 Exam 2

Please read the following instructions. For the following exam you are free to use any papers or books you like, but no calculators or computers. Please turn in **exactly five** problems. You must do problems 1,2,3, and two more chosen from the remainder. Please write your answers on separate paper, make clear what work goes with which problem, and put your name or initials on every page. You have 75 minutes. Each problem will be graded on a 5-10 scale (as your quizzes), for a total score between 25 and 50. This will then be doubled for your exam score.

Turn in problems 1, 2, 3:

1. Set $A = \begin{pmatrix} 3 & 1 & -1 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. Find J_A, the Jordan canonical form for A.

2. Let $A \in M_n(\mathbb{C})$, and let $\lambda \in \mathbb{C}$ be fixed. Prove that the following are equivalent:
(1) λ is NOT an eigenvalue of A;
(2) $w_k(A, \lambda) = 0$ for all $k \geq 1$; and
(3) $w_1(A, \lambda) = 0$.

3. Let $B \in M_n(\mathbb{C})$ be invertible. Suppose that there is some $A \in M_n(\mathbb{C})$ with $B = A^\star A$. Prove there exists a unique $L \in M_n(\mathbb{C})$, such that L is lower triangular, has positive real diagonal entries, and $B = LL^\star$.

Turn in exactly two more problems of your choice:

4. Find the rational canonical form and the real Jordan canonical form of $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

5. For each $n \geq 4$, find nilpotent matrices $A, B \in M_n$ that are not similar to each other, but have the same minimal polynomial.

6. Let $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and set $W_J = \begin{bmatrix} \lambda I_n & I_n \\ 0 & \lambda I_n \end{bmatrix}$. Prove that $A \in M_{2n}$ commutes with W_J if and only if $A = \begin{bmatrix} B & C \\ 0 & B \end{bmatrix}$ for some matrices $B, C \in M_n$.

7. Let $A \in M_n$. Prove that A commutes with $J_n(0)$ if and only if A is an upper triangular Toeplitz matrix.